Applied Mathematics and Computation 331 (2018) 96–111

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A generalized Laguerre spectral Petrov–Galerkin method for the time-fractional subdiffusion equation on the semi-infinite domain

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ARTICLE INFO

Keywords:
Generalized associated Laguerre function
Generalized Laguerre functions
Time-space spectral Petrov–Galerkin method
Time-fractional subdiffusion equation
Semi-infinite domain

ABSTRACT

In this paper, a generalized Laguerre spectral Petrov–Galerkin method is proposed to solve a class of time-fractional subdiffusion equations on the semi-infinite domain. We use the generalized associated Laguerre functions of the first kind and the generalized Laguerre functions as basis functions in time and space directions separately. The respective projection error estimates can be obtained as well. We derive the projection error estimates of the solutions in time-space directions. The approximation results of the fully discrete spectral scheme are introduced in this paper. Some numerical results are presented to illustrate the efficiency of this method.

1. Introduction

We consider here the time-fractional subdiffusion equation (TFSDE) on the semi-infinite domain.

$$\begin{cases} {}_0^C\partial_t^\beta u(x,t) - \partial_x^2 u(x,t) + u(x,t) = f(x,t), x, t \in \Lambda, \\ u(0,t) = 0, \ \lim\limits_{x\to\infty} u(x,t) = 0, \\ u(x,0) = \phi(x), \end{cases} \tag{1.1}$$

where $0 < \beta < 1$, Λ stands for the interval $(0, +\infty)$. The left-sided Caputo fractional derivative of order β with respect to time t is denoted by ${}_0^C\partial_t^\beta$ which will be introduced in the next section. ∂_x^k stands for the usual derivative, $f(x, t)$ and $\phi(x)$ are given functions. For the sake of clarity, we only consider the homogeneous initial value condition $\phi(x) = 0$, the inhomogeneous case can be transformed into homogeneous condition by using the homogeneous transformation.

The time-fractional diffusion equations can be divided into the time-fractional subdiffusion and superdiffusion equations. Roughly speakin g, compared with the superdiffusion phenomena, subdiffusion phenomena are more widespread in the real world. Subdiffusion processes which are commonly described by the continuous time random walks which are known as non-Brownian subdiffusion [1]. It may appear in some viscoelastic media. The presence of traps in chaotic systems are also the reasons of the anomalous subdiffusion [2]. Meanwhile, the charge carrier motion in amorphous semiconductors and nuclear magnetic resonance also include subdiffusion phenomena [3]. In most cases, the TFSDEs are considered on the finite domain. However, when the fractional subdiffusion phenomena appear in earthquake engineering and underwater acoustic fields, we need to consider them on the semi-infinite domain [4].

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https://doi.org/10.1016/j.amc.2018.02.050
0096-3003/

One of the difficulties when dealing with the TFSDEs is that the fractional derivatives involve non-local operators. The solutions o f TFSDEs are usually singular near the boundaries. One significant attempt to circumvent this barrier is the use of the spectral method. Jiao et al. [5], Chen et al. [6], Tang et al [7]. Zhang et al. [8], Gassner et al. [9], Hammad and El-Azabb [10], Owolabi [11,12] have made some advanced works in the spectral method field. Generally speaking, if th e basis functions are chosen appropriately, it may create a sparse linear system and match the singular phenomenon near the boundaries. To the best of our knowledge, some remarkable works on spectral method have been done on the finite domains. Dehghan and Abbaszadeh [13], Zheng et al. [14], Donatelli et al. [15], Chen et al. [16], Pindza and Owolabi [17] and Li and Xu [18] proposed some numerical methods combined the spectral method with some other effective numerical methods like the finite element method, finite difference method, etc. to solve the TFSDEs on the finite domains. Meanwhile, Bhrawy [19,20], Zhang and Yi [21], Li et al. [22], Sohrabi, Ranjbar and Saei [23], Shindin et al. [24], Owolabi [25,26] also made great contributions to this field by introducing some generalized Jacobi polynomials/functions. Another difficulty is compared with the TFSDEs on the finite domains, we have fewer basis functions when dealing with the semi-infinite problems. Most of the works are based on the Laguerre polynomials. To date, the study of the spectral method for the fractional diffusion problems on the semi-infinite domain has made certain achievements. Bhrawy et al. [27–31] proposed some effective spectral methods based on the generalized Laguerre polynomials. In [4], they solved some kinds of TFSDEs on the infinite domains. Baleanu et al. [32] used the modified generalized Laguerre spectral method for the fractional problems on the half line. Meanwhile, Khader [33,34], Khosravian et al. [35], Zhokh and Trypolskyi [36], Khosravian et al. [37], Owolabi [38,39], Hao et al. [40], Wei [41], Varatharajan and DasGupta [42] also made lots of significant contributions to this field.

The main purpose of this paper is to use a generalized Laguerre spectral Petrov–Galerkin method to solve a class of TFSDEs on the semi-infinite domain. The GALFs-1 and GLFs are introduced as basis functions in time and space directions separately. In time direction, the GALFs-1 can treat the singularity of the underlying solution. Some remarkable approximate results can be achieved in time direction. In space direction, expected results can be achieved by using the GLFs as basis functions. Therefore, the time-space approximate results can be obtained as well. At the end of this paper, we present some numerical results to validate this method.

The outline of this paper is organized as follows. In Section 2, we recall some definitions, notations and properties. In Section 3, the respective projection error estimates of the time and space directions are obtained. The projection error estimates of time–space directions are also derived. At the end of this section, the implementation technique of the method is presented. Some examples are shown to illustrate the efficiency of this method in Section 4. Finally, some conclusions and future work are given in Section 5.

2. Preliminaries

In this section, we briefly review several definitions and notations that are pertinent to the forthcoming development. We recall the definition of the GALFs-1, and some useful properties of GALFs-1 are proposed. As we know, the GALFs-1 play important roles in solving the fractional problems on the semi-infinite domain.

2.1. Some definitions and notations

Throughout this paper, we first recall some definitions and notations. Let $\mathbb{N}$ and $\mathbb{N}_0$ stand for the positive integers and $\{0\}\cup\mathbb{N}$, respectively. Define $(u,v)_{\chi,\Lambda} := \int_\Lambda uv\chi d\Lambda$ is the inner product of $L^2_\chi(\Lambda)$ and $\|\cdot\|_{\chi,0,\Lambda}$ is the norm of it. For a nonnegative real number s, $H^s_\chi(\Lambda)$ and $H^s_{0,\chi}(\Lambda)$ are denoted as the real Sobolev spaces with respect to weight functions χ corresponding to norm $\|\cdot\|_{H^s_\chi(\Lambda)} := \|\cdot\|_{\chi,s,\Lambda}$.

It is well known that $H^s_{0,\chi}(\Lambda)$ is the closure of $C^\infty_{0,\chi}(\Lambda)$ with respect to norm $\|\cdot\|_{\chi,s,\Lambda}$, and $C^\infty_{0,\chi}(\Lambda)$ is the space of smooth functions with compact support in Λ. We define a space which is motivated by $C^\infty_{0,\chi}(\Lambda)$ as follows.

$$ {}_0C^\infty_\chi(\Lambda) := \{v \mid v\in C^\infty_\chi(\Lambda),\ \forall T>0,\ v \text{ has compact support in } (0,\mathrm{T}]\}. $$

The space ${}_0H^s_\chi(\Lambda)$ is denoted to be the closure of ${}_0C^\infty_\chi(\Lambda)$ with respect to norm $\|\cdot\|_{\chi,s,\Lambda}$.

Let X be a Sobolev space with norm $\|\cdot\|_X$. We have

$$ H^s_\chi(\Lambda;X) := \left\{v \mid \|v(\cdot,t)\|_X \in H^s_\chi(\Lambda)\right\} $$

endowed with the norm

$$ \|v\|_{H^s_\chi(\Lambda;X)} := \|\,\|v(\cdot,t)\|_X\|_{\chi,s,\Lambda}\ . $$

Similarly, we have

$$ H^s_{0,\chi}(\Lambda;X) := \left\{v \mid \|v(\cdot,t)\|_X \in H^s_{0,\chi}(\Lambda)\right\},\quad {}_0H^s_\chi(\Lambda;X) := \left\{v \mid \|v(\cdot,t)\|_X \in {}_0H^s_\chi(\Lambda)\right\},\ s>0 $$

endowed with the norm

$$ \|v\|_{H^s_{0,\chi}(\Lambda;X)} := \|\,\|v(\cdot,t)\|_X\|_{\chi,s,\Lambda},\quad \|v\|_{{}_0H^s_\chi(\Lambda;X)} := \|\,\|v(\cdot,t)\|_X\|_{\chi,s,\Lambda}\ . $$

In particular, if X is $H^\sigma_{\chi_1}(\Lambda)$ or $H^\sigma_{0,\chi_1}(\Lambda)$ for $\sigma\geq 0$, the norm of the space $H^s_\chi(\Lambda;X)$ may be denoted by $\|\cdot\|_{\chi_1,\chi,\sigma,s,\Lambda}$. For convenience, the domain symbol Λ may be dropped from the notations directly.