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The least eigenvalue of graphs whose complements have only two pendent vertices[☆]

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ABSTRACT

Let G be a simple graph and $A(G)$ be the adjacency matrix of G. The eigenvalues of $A(G)$ are referred to as the eigenvalues of G. In this paper, we characterize the graphs with the minimal least eigenvalue among all graphs whose complements are connected and have only two pendent vertices.

1. Introduction

Let $G=(V,E)$ be a simple graph of order n with vertex set $V=V(G)=\{v_1,v_2,\dots,v_n\}$ and edge set $E=E(G)$. For any $u,v\in V(G)$, if there is a path which connects u and v, G is called a connected graph. The complement of G is denoted by $G^c:=(V,\,E^c)$, where $E^c:=\{xy\colon x\in V,\ y\in V,\ x\neq y,\ xy\notin E\}$.

The degree matrix of G is denoted by $D(G)=\mathrm{diag}(d_G(v_1),d_G(v_2),\dots,d_G(v_n))$, where $d_G(v)$ denotes the degree of a vertex v in the graph G. The adjacency matrix of the graph G is defined to be a matrix $A(G)=[a_{ij}]$ of order n, where $a_{ij}=1$ if v_i is adjacent to v_j, and $a_{ij}=0$ otherwise. The eigenvalues of $A(G)$ are referred to the eigenvalues of G. The signless Laplacian matrix of G is defined to be $Q(G)=D(G)+A(G)$. Since $A(G)$ and $Q(G)$ are symmetric and real, their eigenvalues are real and can be arranged. Let the eigenvalues of $A(G)$ be arranged as: $\lambda_n(G)\geqslant\lambda_{n-1}(G)\geqslant\cdots\geqslant\lambda_1(G)$. The largest eigenvalue of $A(G)$ is called the spectral radius of G. The least eigenvalue $\lambda_1(G)$ is denoted by $\lambda_{\min}(G)$, and the corresponding eigenvectors are simply called the first eigenvectors.

There are a lot of research results on the (signless Laplacian) spectral radius of the graph. However, relative to the (signless Laplacian) spectral radius, the least (signless Laplacian) eigenvalue was received less attention; see e.g. [1–5,7–16]. Especially, when the structures of graphs are very complicated, but the structures of their complements are simple, we naturally think whether we can study the (signless Laplacian) minimum eigenvalue of the graphs starting from structure of their complements; see e.g [5,8,12,14,16,17].

For convenience, a graph is called minimizing in a certain class if its least eigenvalue attains the minimum among all graphs in this class. In [17], Yu et al. study the minimum eigenvalues of the graphs whose complements are connected and have pendant vertices, and give the corresponding minimizing graph, whose complement is connected and has only one pendent vertex. So, a problem arises naturally what is the minimizing graph among all graphs of order n whose comple-

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ments are connected and have only two pendent vertices. In this paper we will address ourselves to this problem, and study the least eigenvalue of graphs from their complements.

2. Preliminaries

A vector $X \in R^n$ is said to be defined on the graph G, if there is a $1-1$ map φ from $V(G)$ to the entries of X such that $\varphi(u) = X_u$ for each $u \in V(G)$. If X is an eigenvector of $A(G)$, then it is naturally defined on $V(G)$, i.e X_u is the entry of X corresponding to the vertex u. Thus, it is easy to find that

$$X^T A(G) X = 2 \sum_{uv \in E(G)} X_u X_v, \tag{2.1}$$

and when λ is an eigenvalue of G corresponding to the eigenvector X if and only if $X \neq 0$, we obtain the following eigen-equation of the graph G:

$$\lambda X_v = \sum_{u \in N_G(v)} X_u, \text{ for each } v \in V(G), \tag{2.2}$$

where $N_G(v)$ is the set of neighbors of v in G. In addition, for an arbitrary unit vector $X \in R^n$,

$$\lambda_{\min}(G) \leqslant X^T A(G) X, \tag{2.3}$$

with equality if and only if X is a first eigenvector of G.

It is easily seen that $A(G^c) = J - I - A(G)$, where J and I are all-ones matrix and identity matrix of size same as of the adjacency matrix $A(G)$, respectively. Thus, for any vector $X \in R^n$:

$$X^T A(G^c) X = X^T (J - I) X - X^T A(G) X. \tag{2.4}$$

Lemma 2.1 [6]**.** *Let A be a real symmetric $n \times n$ matrix, B be the $m \times m$ principal submatrix of A, and $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$, $\lambda_1(B) \geq \lambda_2(B) \geq \cdots \geq \lambda_m(B)$ are respectively the eignvalues of A and B, then $\lambda_{n-m+i}(A) \leq \lambda_i(B) \leq \lambda_i(A)$ for $i = 1, 2, \ldots, m$.*

3. Main results

In this section, we sometimes use MATLAB to calculate.

Let $H_1(p, q)$ be the special graph of order n $(n = p + q + 2, p \geqslant q \geqslant 0)$ with two pendent vertices v_4, v_6, which adjacent v_3, v_5 respectively, see Fig. 3.1. In particular, when $q = 0$, $v_1 = v_2 = v_5$; when $q = 1$, $v_2 = v_5$.

Lemma 3.1. *Let p, q and n be nonnegative integers such that $p \geq q \geq 0$, $n \geq 17$ and $p + q = n - 2$, we have*

$$\lambda_{\min}(H_1(p, q)^c) \geq \lambda_{\min}(H_1(\lceil (n-2)/2 \rceil, \lfloor (n-2)/2 \rfloor)^c),$$

the equality holds if and only if $p = \lceil (n-2)/2 \rceil, q = \lfloor (n-2)/2 \rfloor$.

Proof. Since $K_2 \subset H_1(p, q)^c$, $\lambda_{\min}(K_2) = -1$. By Lemma 2.1:

$$\lambda_{\min}(H_1(p, q)^c) \leq -1. \tag{3.1}$$

Let X be the first eigenvector of $H_1(p, q)^c$. □

Case 1: $q = 0$. By equalities (2.2) and (3.1), all the vertices in $V(K_p)$ except v_1, v_3 have the same values given by X, say X_1; the vertices v_1, v_3 have the same values given by X, say X_2; the vertices v_4, v_6 have the same values given by X, say X_3. Denote $\lambda_{\min}(H_1(n-2, 0)^c) = \lambda$, also by equality (2.2), we have

$$\begin{cases} \lambda X_1 = 2X_3, \\ \lambda X_2 = X_3, \\ \lambda X_3 = (n-4)X_1 + X_2 + X_3. \end{cases}$$

The above equations are transformed into a matrix equation $(B - \lambda I)X' = 0$, where $X' = (X_1, X_2, X_3)^T$,

$$B = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ n-4 & 1 & 1 \end{pmatrix}.$$

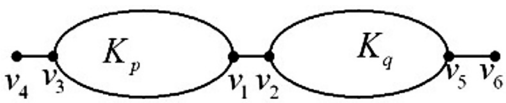

Fig. 3.1. The graph $H_1(p, q)$.