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Stokes' second problem of viscoelastic fluids with constitutive equation of distributed-order derivative

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ABSTRACT

The steady-state periodic flow of Stokes' second problem for viscoelastic fluids with constitutive equation in terms of the distributed-order derivative was considered. The distributed-order derivative involves an integration with respect to the order of fractional derivative, and the order is associated with a weight function $p(\alpha)$. With a general weight function $p(\alpha)$, the flow velocity was obtained. The amplitude, the penetration depth and the wavelength were given analytically. Results of Newtonian fluid and single fractional constitutive equation were derived as special cases of weight function $p(\alpha)$. Also we considered other three cases of weight function $p(\alpha)$: linear combination of Dirac delta functions, constant and parameterized exponential function.

1. Introduction

Viscoelastic fluids belong to non-Newtonian fluid, possess both viscosity and elasticity. Such fluids deform or flow under the action of external forces. When the external force is removed, these deformations can recover or partly recover. Viscoelastic fluids represent complex rheological properties including the phenomena of rod-climbing, die-swell and complicated boundary-layer theory [1,2]. Problems on flows of viscoelastic fluids often occur in chemical and food industries, biology, medicine and polymer processing.

We recall the steady-state periodic flow of the Stokes' second problem, which is a typical flow in fluid mechanics. Homogeneous incompressible fluid occupies a semi-infinite domain, which is above an infinitely extended oscillating plate in the horizontal (x, z) coordinate plane that undergoes harmonic vibration parallel to itself. For the steady-state periodic flow of Newtonian fluid, the dimensionless model is given as

$$\frac{\partial u}{\partial t} = \eta \frac{\partial^2 u}{\partial y^2}, \quad \eta > 0, \quad y > 0, \tag{1}$$

$$u(y,t) = \cos(\omega t) \text{ at } y = 0, \tag{2}$$

$$u(y,t) \to 0 \text{ as } y \to +\infty, \tag{3}$$

where $u = u(y, t)$ is the flow velocity and ω is the vibration frequency of the plate. The solution of the problem is [3,4]

$$u(y,t) = \exp\left(-y\sqrt{\frac{\omega}{2\eta}}\right)\cos\left(\omega t - y\sqrt{\frac{\omega}{2\eta}}\right). \tag{4}$$

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Such problem for different fluid models was considered, including the Maxwell viscoelastic fluid [5], the thixotropic or antithixotropic fluid [6].

In recent decades, the fractional calculus has been introduced to the study of viscoelastic theory, anomalous diffusion, control theory, image processing, dynamical system, etc [7–18]. Scott-Blair [7,8] put forward a constitutive relation with fractional derivative for viscoelastic body based on the idea that the mechanical properties of such material are between elastic solids and viscous fluids. New terminologies “Scott-Blair model” [14,19] and “spring-pot” [20] appeared to describe the constitutive element of viscoelastic body. Based on fractional constitutive relation of viscoelastic body, experiment and parametric simulation were conducted [21–23], and vibration dynamics and stability theory for damped system were considered [24–29].

We recall the basic concepts in the fractional calculus. Let $f(t)$ be piecewise continuous on $(a, +\infty)$, then the Riemann–Liouville fractional integral of $f(t)$ of order μ is defined as [12,30,31]

$$ {}_a\mathrm{I}_t^{\mu} f(t) = \int_a^t \frac{(t-s)^{\mu-1}}{\Gamma(\mu)} f(s)ds \text{ for } \mu > 0, \tag{5} $$

and ${}_a\mathrm{I}_t^{\mu} f(t) = f(t)$ for $\mu = 0$, where $\Gamma(\cdot)$ is Euler's gamma function.

The Riemann–Liouville and Caputo fractional derivatives of $f(t)$ of order α are defined as [12,30,31]

$$ {}_a\mathrm{D}_t^{\alpha} f(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} {}_a\mathrm{I}_t^{n-\alpha} f(t),\ 0 \le n-1 < \alpha < n,\ \ n \in \mathbb{N}^+, \tag{6} $$

and

$$ {}_a^*\mathrm{D}_t^{\alpha} f(t) = {}_a\mathrm{I}_t^{n-\alpha} f^{(n)}(t),\ 0 \le n-1 < \alpha < n,\ \ n \in \mathbb{N}^+, \tag{7} $$

respectively.

For sake of considering the steady-state periodic flow, we take the lower limit a in the definitions to be $-\infty$ in this work. Thus the two definitions of fractional derivatives are consistent with one another [12]. We denote them as ${}_{-\infty}\mathrm{D}_t^{\alpha} f(t)$.

The Scott-Blair model for the constitutive equation of viscoelastic body for the shear stress τ and the shear strain ϵ is [12,14,20,32,33]

$$ \tau = \eta\sigma^{\alpha}\, {}_{-\infty}\mathrm{D}_t^{\alpha}\epsilon = \eta\sigma^{\alpha}\, {}_{-\infty}\mathrm{I}_t^{1-\alpha}\dot{\epsilon}, \quad 0 < \alpha \le 1, \tag{8} $$

where η is the shear modulus and σ is the relaxation time.

In [34], the Stokes' second problem for viscoelastic fluid with the fractional constitutive relation was considered, in the dimensionless form, as

$$ \frac{\partial u}{\partial t} = \eta\, {}_{-\infty}\mathrm{I}_t^{1-\alpha} \frac{\partial^2 u}{\partial y^2}, \quad y > 0, \quad 0 < \alpha \le 1, \tag{9} $$

$$ u(y,t) = \cos(\omega t) \text{ at } y = 0, \tag{10} $$

$$ u(y,t) \to 0 \text{ as } y \to +\infty. \tag{11} $$

The solution of the problem is

$$ u(y,t) = \exp\left(-y\sqrt{\frac{\omega^{2-\alpha}}{\eta}} \sin\frac{\pi\alpha}{4}\right) \cos\left(\omega t - y\sqrt{\frac{\omega^{2-\alpha}}{\eta}} \cos\frac{\pi\alpha}{4}\right). \tag{12} $$

For the limit case $\alpha \to 0$, Eq. (9) becomes

$$ \frac{\partial u}{\partial t} = \eta\, {}_{-\infty}\mathrm{I}_t^{1} \frac{\partial^2 u}{\partial y^2}, \tag{13} $$

which is equivalent to wave equation, and the steady solution is

$$ u(y,t) = \cos(\omega t - \omega y/\sqrt{\eta}), \tag{14} $$

where $\sqrt{\eta}$ is wave velocity. In fact, the case corresponds to the Hooke's law in Eq. (8) and elastic waves in solid. We note that the boundary condition (11) is not satisfied for the limit case.

In recent years, distributed-order derivative which permits integration with respect to the order of fractional derivatives was proposed to modelling anomalous diffusion and constitutive relation of viscoelastic body [35–41]. The distributed-order and variable-order fractional models in transient sub-diffusion were investigated [42].

In this work, we consider Stokes' second problem for viscoelastic fluids with constitutive relation in terms of distributed-order derivative.