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Local variably scaled Newton basis functions collocation method for solving Burgers' equation

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ABSTRACT

In the present paper, a meshless method which constructs derivatives discretizations based on variably scaled Newton basis functions (VSNBFs) interpolants for localized sets of nodes is provided to solve Burgers' equation in the "native" Hilbert space of the reproducing kernel. Using such formulas for discretizations of PDEs when each internal node considered as a center, leads to a sparse global of system of ordinary differential equations (ODEs) to be formed by a simple assembly of function values at the local system centrepoints. For efficiency and stability reasons, we use the interpolation with variably scaled kernels, which are introduced recently by Bozzini et al. The VSNBFs can be constructed from a symmetric positive definite variably scaled kernel. The main advantage of using the local method is that the overlapping domains of influence result in many small matrices with the dimension of the number of nodes included in the domain of influence for each center node, instead of a large collocation matrix, and hence, sparse global derivative matrices from local contributions. Therefore, the method needs less computer storage and flops. The method is tested on six benchmark problems on Burgers' equation with mixed boundary conditions. Five number of nodes are used in the domain of influence of the local support. The numerical results for different values of Reynolds numbers (*Re*) are compared with analytical solutions as well as some other numerical methods. It is shown that the proposed method is efficient, accurate and stable for flow with reasonably high values of *Re*. Numerical results also show that using variably scaled kernels leads to results that are better than using constant scaled kernels, with respect to both stability and error.

1. Introduction

Unlike traditional numerical methods in solving PDEs, *meshless methods* [5] need no mesh generation. Collocation methods are truly meshless and simple to program, and they allow various approaches to solve PDEs. Taking translates of kernels as trial functions, meshless collocation in unsymmetric and symmetric form dates back to [17,18,24] and has been proved to be highly successful, since the arising linear systems are easy to generate and allow good accuracy at low computational cost. In addition, it was recently proved [48] that symmetric collocation [17,18] using kernel basis is optimal along all linear PDE solvers using the same input data. This motivates the use of kernels to solve PDEs. An overview of kernel methods before 2006 is presented in [51], while their recent variations are in [2,3,21,28,39–42] and the references therein.

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Fig. 1. The uniform node arrangement and the schematics of the local domains of influence in the interior, near boundary and corner points using $n_s = 5$ in one dimension, where $\times$ denotes the center node.

The choice of a scale or shape parameter of a kernel or radial basis function (RBF) is a problem that has been around for more than two decades (see e.g. [6,33–37,46,53]). There are purely experimental reports on the behavior of kernel-based methods under scaling, and there are optimization techniques that attempt to find a good compromise between bad condition and good reproduction quality. Recently, Bozzini et al. [6] developed the case of varying scales by letting the scale parameter be an additional coordinate [6]. This allows varying scales without leaving the firm grounds of kernel–based interpolation. It turns out that this approach can be fully understood as the standard constant scaled method applied to a certain sub-manifold of $\mathbb{R}^{d+1}$.

It is well known that the representations of kernel-based approximants in terms of the standard basis of translated kernels are notoriously unstable due to leading an ill-conditioned kernel matrix when the separation distance is small. The Newton basis [43] with a recursively computable set of functions, which vanish at increasingly many data points, turns out to be more stable. It is orthonormal and complete in the native Hilbert space, if infinitely many data locations are reasonably chosen. Recently, an adaptive calculation of Newton basis arising from a pivoted Cholesky factorization, which is computationally cheap, has been introduced [44].

For time-dependent PDEs, meshless kernel-based methods are based on a fixed spatial interpolation, but since the coefficients are time-dependent, one obtains a system of ODEs. This is the well-known *method of lines*, and it turned out to be accurate in several problems [11,12,14,54].

Among meshless kernel-based methods, the RBF collocation method has been vastly applied to solve various kinds of physical problems [16,18,25,26]. Since the resultant matrix in RBF collocation method is constructed by taking direct collocation of nodes in the whole domain, the solution is obtained from solving a dense, and hence ill-conditioned, collocation matrix. This limits the applicability of the RBF collocation method to solve large scale problems. In the past, various techniques such as Wendland's Compactly Supported RBF [56], preconditioning technique by Ling and Kansa [31], adaptive greedy algorithm by Ling and Schaback [32] and domain decomposition method by Li and Hon [30] have been proposed to tackle this ill-conditioning problem.

More recently, the local RBF collocation method applying collocation separately on each overlapping sub-domain of the whole domain, has been devised [29,47,52,57]. This localization dramatically reduces the denseness of the resultant collocation matrix but at the expense of solving many small sub-matrices with the dimension of the number of nodes included in the domain of influence for each center node.

In the present paper, a way of using VSNBFs as the basis for PDE's solvers is presented, its essence being constructing approximate formulas for derivatives discretizations based on VSNBFs interpolants with local supports. For efficiency and stability reasons, the interpolation with variably scaled kernels introduced by Bozzini et al., is used. The VSNBFs can be constructed from a symmetric positive definite variably scaled kernel. We construct approximations to derivatives at each node based on VSNBFs interpolants for a limited number of neighbor nodes forming a stencil (see Fig. 1). Then the values of derivatives of the unknown function at each node can be written in terms of nodal values at the interpolation nodes which lie inside the influence domain. After applying collocation separately on each overlapping sub-domain of the whole domain, the PDE governing operator and any required boundary operators are already enforced. This leads to a sparse global of system of ODEs to be formed by a simple assembly of function values at the local system centrepoints. Since the derivative is a local property, intuitively it seems reasonable that it can be accurately approximated without using information from the entire domain. So the local method can have accuracy comparable to the global method while also enjoying both considerably smaller storage requirements and a smaller flop count than the global method. In principle, it is possible to use both uniform or random nodal arrangements in the method implementation due to the meshless character of the method. But one advantage of using uniform nodal arrangement is that small spatial derivatives and collocation matrices (of the size of sub-domain which is 5×5 in the present case) corresponding to each stencil needs to be computed only one time. This saves considerable amount of CPU time as well as memory. Chebyshev nodes are the most important set of non-uniform points for polynomial interpolation. But, if kernel-based methods are used, they lead to a very large condition number in the kernel matrices, no matter which kernel is chosen [6]. To cope with this, we transform non-uniform Chebyshev nodes into equidistant nodes on the circle by a variably scaled kernel with the scale function $c(x) = \sqrt{1 - x^2}$. This provides a stabilized calculation of weights. So the proper use of the variably scaled kernels can lead to more stable results.

The Burgers' equation is the simplest nonlinear model equation for diffusive waves in fluid dynamics. Burgers' equation arises in many physical problems including one-dimensional turbulence, sound waves and shock waves in a viscous medium, waves in fluid filled viscous elastic tubes, and magneto-hydrodynamic waves in a medium with finite electrical conductivity. The Burgers' equation is similar to the one-dimensional Navier–Stokes equation without the stress term. A great deal of effort has been expended in the last few years to compute the numerical solutions of the Burgers' equation for the small and large values of the kinematic viscosity (see e.g. [1,4,8–10,15,19,20,22,23,27,38,45,55,58]). Our proposed method is also tested on six benchmark problems on Burgers' equation with mixed boundary conditions. Five number of nodes are used in the domain of influence of the local support. The numerical results for different values of *Re* are compared with analytical