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# Evaluation of mixed Crank–Nicolson scheme and Tau method for the solution of Klein–Gordon equation

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ABSTRACT

Numerical method based on the Crank–Nicolson scheme and the Tau method is proposed for solving nonlinear Klein–Gordon equation. Nonlinear Klein–Gordon equation is reduced by Crank–Nicolson scheme to the system of ordinary differential equations then Tau method is used to solve this system by using interpolating scaling functions and operational matrix of derivative. The order of convergence is proposed and some numerical examples are included to demonstrate the validity and applicability of the technique. The method is easy to implement and produces accurate results.



## 1. Introduction

The nonlinear Klein–Gordon equation is used to model many nonlinear phenomena. It represents the equation of motion of a quantum scalar or a pseudo-scalar field, which is a field whose quanta are spinless particles. Such a problem appears naturally in the study of some nonlinear dynamical problems of mathematical physics, such as radiation theory, general relativity, the scattering and stability of kinks, vortices and other coherent structures [11]. At first, this equation which describes relativistic electrons was described by two physicists Oskar Klein and Walter Gordon in 1926. The Klein–Gordon equation correctly describes the spinless pion, a composite particle where the Dirac equation describes the spinning electron. Schrodinger proposed the Klein–Gordon equation as a quantum wave equation in his search for an equation describing de Broglie waves in 1927. The fine structure of hydrogen atom is predicted incorrectly by this equation. A non-relativistic approximation that predicts the Bohr energy levels of hydrogen without fine structure is submitted by Schrodinger instead of his equation. A linear and nonhomogeneous Klein–Gordon equation has the form [6,33]

$$(\Box + b)u = f(x,t),$$

where $\Box$ is the d'Alembert operator, defined by $\Box = (\frac{\partial^2}{\partial t^2} - a^2\nabla^2)$. Also a nonlinear and nonhomogeneous Klein–Gordon equation is given by [10,11,13,26,36]

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - g(u) = f(x,t).$$

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This equation with $f(x,t)=0$ is encountered in quantum field theory and a number of applications and is referred to as the nonlin ear Klein–Gordon equation so that some special cases are obtained by selecting the nonlinear force $g(u)$ as

- Klein–Gordon equation with a power-law nonlinearity 1 [33], $g(u)=\beta u+\gamma u^m$.
- Klein–Gordon equation with a power-law nonlinearity 2 [33], $g(u)=\beta u^m+\gamma u^{2m-1}$.
- Modified Liouville equation [5,33], $g(u)=\gamma e^{\kappa u}$.
- Klein–Gordon equation with a exponential nonlinearity [33], $g(u)=\beta e^{\kappa u}+\gamma e^{2\kappa u}$.
- Sine-Gordon equation [16,33], $g(u)=\gamma sinh(\lambda u)$.
- Sine-Gordon equation [3,33], $g(u)=\gamma sin(\lambda u)$.

With reference to the solution of this equation, we can see many published papers: The differential transform method is proposed for solving linear and nonlinear Klein–Gordon equation [36]. Authors of [26] studied the nonlinear Klein–Gordon equation with quadratic and cubic nonlinearities and proposed a numerical scheme to solve this equation using collocation and finite difference-collocation methods. In [26] b-spline scaling function is used for solving this equation. Numerical solution based on finite difference and collocation method of the nonlinear Klein–Gordon equation using radial basis functions is proposed in [13]. Four finite difference schemes for approximating the nonlinear Klein–Gordon equation have been presented in [25]. In [23] Li and Guo proposed a Legendre pseudospectral scheme for solving initial value problem of the nonlinear Klein–Gordon equation. Their numerical solution keeps the discrete conservation also the stability and convergence are investigated in this paper. Nonlinear Klein–Gordon equation on an unbounded domain is studied in [17]. In [17] Split local absorbing boundary (SLAB) conditions are obtained by the operator splitting method, then the original problem is reduced to an initial boundary value problem on a bounded computational domain, which can be solved by the finite difference method. In [8], Adomians decomposition scheme is used as an alternate method for solving this equation. In [43], Wazwaz proposed the modified decomposition method for analytic treatment of nonlinear differential equations. The modified method accelerates the rapid convergence of the series solution and provides the exact solution by using few iterations only without any need to the so-called Adomian polynomials. In [43], nonlinear Klein–Gordon equation is solved by the modified decomposition method. In [24], Li and Quoc studied a formalism deriving systematically invariant, finite difference algorithms for nonlinear Klein–Gordon equation. The papers by [21,22,30,38] may be consulted on the progress made during the last 20 years.

In this work, the nonlinear and nonhomogeneous Klein–Gordon equation is assumed as

$$u_{tt}-u_{xx}+g(u)=f(x,t), x\in\Omega=[0,1]\subset\mathbb{R}, t\geq 0, \tag{1.1}$$

with initial conditions

$$u(x,0)=f_1(x),\quad u_t(x,0)=f_2(x),\quad x\in\Omega, \tag{1.2}$$

and boundary condition

$$u(x,t)=p(x,t),\quad x\in\partial\Omega\quad t\geq 0, \tag{1.3}$$

where $u(x, t)$ is the unknown solution which must be found. $g(u)$ satisfies the Lipschitz condition with respect to $u$

$$|g(u)-g(\hat{u})|\leq\mathfrak{L}|u-\hat{u}|, \tag{1.4}$$

where $\mathfrak{L}$ is Lipschitz constant.

The proposed equation will be reduced by Crank–Nicolson scheme to system of ordinary differential equations then the Tau method will be used to solve this system. Spectral Tau methods are similar to Galerkin methods in the way the differential equation is enforced. However, none of the test functions need to satisfy the boundary conditions. Hence, a supplementary set of equations is used to apply the boundary conditions [4,7]. The Tau method have been extended in several directions such as: systems of linear differential equations [15], non-linear problems [31,32]; partial differential equations [34]; and, in particular, to the numerical solution of non-linear systems of partial differential equations [19], to the numerical solution of Fredholm integro-differential equation and Volterra integral equation [20,42].

In this work, interpolating scaling functions which are introduced by Alpert et al. [1,2] are used. These functions are based on Legendre polynomials [37]. In addition to the simultaneous properties which proposed for multiwavelets, interpolating scaling functions have interpolating property. This feature helps to reduce the time and computational cost. Also operational matrix of derivative is derived in [12,39] which helps to save computing time.

The organization of the paper is as follows, In Section 2, interpolating scaling functions are introduced and the operational matrix of derivative is obtained. In Section 3, we apply the proposed method to solve the nonlinear Klein–Gordon equation. The order of convergence is obtained in Section 4. The results of numerical experiments are presented in Section 5. Section 6 is dedicated to a brief conclusion.

## 2. Interpolation scaling functions

Due to some desirable properties, multi-wavelets have been found to be interesting rather than scaler wavelets [41]. The multi-wavelets used in this paper are Alpert' multi-wavelets [1,2]. In this work, the Alpert' scaling functions known by the name of interpolating scaling functions are applied for solving Klein–Gordon equation. Assume $P_r$ is the Legendre polynomial