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A two level algorithm for an obstacle problem

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ABSTRACT

Due to the inequality feature of the obstacle problem, the standard quadratic finite element method for solving the problem can only achieve an error bound of the form $\mathcal{O}(N^{-3/4+\epsilon})$, N being the total number of degrees of freedom, and $\epsilon > 0$ arbitrary. To achieve a better error bound, the key lies in how to capture the free boundary accurately. In this paper, we propose a two level algorithm for solving the obstacle problem. The first part of the algorithm is through the use of the linear elements on a quasi-uniform mesh. Then information on the approximate free boundary from the linear element solution is used in the construction of a quadratic finite element method. Under some assumptions, it is shown that the numerical solution from the two level algorithm is expected to have a nearly optimal error bound of $\mathcal{O}(N^{-1+\epsilon})$, $\epsilon > 0$ arbitrary. Such an expected convergence order is observed numerically in numerical examples.

1. Introduction

Many problems in physical and engineering sciences are modeled by partial differential equations. However, various more complex physical processes are described by variational inequalities (VIs). Variational inequalities form an important family of nonlinear problems arising in diverse application areas, for example, elastoplasticity, contact mechanics, heat control problems, options pricing problems in finance, Nash-equilibria in management science. Therefore, how to solve variational inequalities efficiently is very attractive to mathematicians, engineers and economists. Variational inequalities of the first kind are closely related to free-boundary problems. The classical formulation of a variational inequality is usually expressed through the presence of an unknown region or boundary. So a variational inequality can be also viewed as a free-boundary problem. Moreover, many free-boundary problems can be reformulated as variational inequalities. The formulation of a variational inequality is advantageous over that of a free-boundary problem, especially for numerical solutions, since in a variational inequality there is no explicit involvement of an unknown region or boundary.

In this paper, we consider an obstacle problem, which is a representative elliptic variational inequality (EVI) of first kind [6]. For more examples of EVIs, we refer the reader to the monograph [3]. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a Lipschitz boundary $\partial\Omega$.

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An obstacle problem. Let $f \in L^2(\Omega)$, and $\psi \in C(\bar{\Omega})$ with $\psi \le 0$ on $\partial\Omega$. The obstacle problem is to find $u \in K$ such that

$$a(u, v-u) \ge (f, v-u)_\Omega \quad \forall v \in K, \tag{1.1}$$

where

$$K = \{v \in H_0^1(\Omega) : v \ge \psi \text{ a.e in } \Omega\} \tag{1.2}$$

is a closed and convex admissible set of the space $H_0^1(\Omega)$, and

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx,$$
$$(f, v)_\Omega = \int_\Omega f v \, dx.$$

The obstacle problem has a unique solution [3]. It arises in a variety of applications, such as the membrane deformation in elasticity theory, and the non-parametric minimal and capillary surfaces as geometrical problems. The elastic-plastic torsion problem and the cavitation problem in the theory of lubrication also can be regarded as obstacle type problems. If the solution has the regularity $u \in H^2(\Omega)$, then it satisfies the relations (see, e.g., [1])

$$-\triangle u \ge f, \quad u \ge \psi, \quad (-\triangle u - f)(u - \psi) = 0 \quad \text{a.e. in } \Omega. \tag{1.3}$$

Therefore, we have the following relations:

$$-\triangle u \ge f \quad \text{in } \Omega^0 = \{x \in \Omega : u(x) = \psi(x)\},$$
$$-\triangle u = f \quad \text{in } \Omega^+ = \{x \in \Omega : u(x) > \psi(x)\}.$$

Regarding the solution regularity of the obstacle problem, we know that $u \in C^{1,1}(\Omega) \cap W^{s,p}(\Omega)$ with $1 < p < \infty$ and $s < 2 + 1/p$ if $f \in W^{1,\infty}(\Omega)$ and $\psi \in C^{1,1}(\Omega) \cap W^{s,p}(\Omega)$; furthermore, if we also have $\partial\Omega \in C^{2,\alpha}$ and $\psi \in W^{2,\infty}(\Omega)$, then $u \in W^{2,\infty}(\Omega)$ up to the boundary $\partial\Omega$ [5,7].

The finite element method is the dominant numerical discretization method for variational inequalities. Optimal convergence order can be reached by the linear elements [4,6,10,11,13–15] under the regularity assumption $u \in H^2(\Omega)$. With the regularity assumptions that $\partial\Omega \in C^{2,\alpha}$, $f \in W^{1,\infty}(\Omega)$, $u \in W^{2,\infty}(\Omega)$, and if there is a negative constant $C(f)$ such that $f \le C(f) < 0$ in Ω, Nochetto proved (see [8,9] and the reference therein) that the distance between the discrete free boundary (by linear finite element method) and the exact free boundary is bounded by $Ch(\log h |u|_{2,\infty,\Omega})^{1/2}$ by using the so-called non-degenerancy property of the obstacle problem [5]. For the quadratic element solutions, an error bound $\mathcal{O}(h^{3/2-\epsilon})$, $\epsilon > 0$ arbitrary, is derived for $H^1(\Omega)$-norm in [12] under regularity assumption that $u \in W^{s,p}(\Omega)$ with $1 < p < \infty$ and $s < 2 + 1/p$. In terms of the total number of degrees of freedom N, the error bound for the linear element solution is $\mathcal{O}(N^{-1/2})$, whereas that for the quadratic element solution is $\mathcal{O}(N^{-3/4+\epsilon})$ for an arbitrarily small $\epsilon > 0$. For variational inequalities, higher order elements do not lead to higher order convergence. Therefore, it is common to use low order elements in solving variational inequalities. In [10], EQ_1^{rot} nonconforming finite element is studied to solve Signorini problem, and the linear convergence order is obtained for quadrilateral meshes satisfying regularity assumption and bi-section condition if $u \in H^{5/2}(\Omega)$. Furthermore, the superconvergence technique is applied to improve the convergence order to $O(h^{3/2})$ for rectangular meshes.

In this paper, we introduce a two level algorithm using both linear and quadratic elements to solve the obstacle problem such that the error bound is expected to be $\mathcal{O}(N^{-1+\epsilon})$ for an arbitrarily small $\epsilon > 0$. In the error analysis for the two level algorithm, we adopt the assumption that $u \in W^{s,p}(\Omega)$ with $1 < p < \infty$ and $s < 2 + 1/p$. Moreover, we assume that $u|_{\Omega^0} \in H^3(\Omega^0)$ and $u|_{\Omega^+} \in H^3(\Omega^+)$, where Ω^0 is the contact area and $\Omega^+ = \Omega \backslash \Omega^0$. This is a reasonable assumption. In the contact area, $u = \psi$, so $u|_{\Omega^0} \in H^3(\Omega^0)$ is just the assumption $\psi|_{\Omega^0} \in H^3(\Omega^0)$, which is implied by $\psi \in H^3(\Omega)$. $u|_{\Omega^+} \in H^3(\Omega^+)$ can be considered as the solution of a Poisson equation with the free-boundary as the Dirichlet boundary. The error bound is proved under some assumption on the behavior of the numerical solution. The idea of the algorithm is outlined as follows. First, solve the obstacle problem with linear elements on a quasiuniform mesh $\mathcal{T}_h$, and identify free-boundary elements. Then refine the free-boundary elements into elements with mesh size $h_* = O(h^{4/3})$ to obtain a new mesh. Finally, we solve the obstacle problem on this new mesh with the quadratic elements. For the numerical analysis and algorithm implementation, we only consider the two dimensional case, although the discussion can be extended to the 3-D case without problem.

The rest of the paper is organized as follows. In Section 2, we introduce the two level algorithm. In Section 3, we derive a priori error estimates for this algorithm. In Section 4, we present numerical examples to provide numerical evidence of the error bound.

2. A two level algorithm

We assume Ω is a polygonal domain. For a subdivision $\mathcal{T}_h$ of $\overline{\Omega}$ into triangles $\{T\}$, let $h_T = \text{diam}(T)$ and $h = \max\{h_T : T \in \mathcal{T}_h\}$. All the subdivisions, including the refined meshes, are constructed so that the minimal angle condition is satisfied. Given a bounded domain $D \subset \mathbb{R}^2$ and a positive integer m, $H^m(D)$ is the Sobolev space with the corresponding usual norm and semi-norm, which are denoted respectively by $\|\cdot\|_{m,D}$ and $|\cdot|_{m,D}$. For convenience, we use $\|\cdot\|_m$ and $|\cdot|_m$ when $D = \Omega$.

We introduce the following two level quadratic finite element method for the obstacle problem: