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Hilfer fractional stochastic integro-differential equations

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ABSTRACT

In this paper, we investigate the existence of mild solutions of Hilfer fractional stochastic integro-differential equations with nonlocal conditions. The main results are obtained by using fractional calculus, semigroups and Sadovskii fixed point theorem. In the end, an example is given to illustrate the obtained results.

1. Introduction

It is well-known that the fractional derivatives are valuable tools for the description of memory and hereditary properties of various materials and processes, which integer order derivatives cant characterize. Many problems in various fields can be described by fractional calculus such as material sciences, mechanics, wave propagation, signal processing, system identification and so on(see [1–4]). The fractional differential equations have gained considerable importance during the past three decades. Hence, the theory of fractional differential equations has emerged as an active branch of applied mathematics. It has been used to construct many mathematical models in various fields, such as physics, chemistry, viscoelasticity, electrochemistry, control, porous media, electromagnetic and polymer rheology, etc. The recent works on the theory and application of fractional differential equations, we refer to the monographs [5–9]. Moreover, stochastic perturbation is unavoidable in nature and hence it is important and necessary to consider stochastic effect into the investigation of fractional differential equations (see [10–13]). Hilfer proposed a generalized Riemann–Liouville fractional derivative for short, Hilfer fractional derivative, which includes Riemann–Liouville fractional derivative and Caputo fractional derivative (see [2,14]). Subsequently, many authors studied the fractional differential equations involving Hilfer fractional derivatives (see [15–19]).

In this paper, we study the existence of mild solutions of Hilfer fractional stochastic integro-differential equations of the form

$$\begin{cases} D_{0+}^{\nu,\mu}[x(t)+F(t,v(t))]+Ax(t)=\int_0^t G(s,\eta(s))d\omega(s), \quad t\in J:=(0,b], \\ I_{0+}^{(1-\nu)(1-\mu)}x(0)-g(x)=x_0, \end{cases} \tag{1.1}$$

where $(t,v(t))=(t,x(t),x(b_1(t)),\dots,x(b_m(t)))$ and $(t,\eta(t))=(t,x(t),x(a_1(t)),\dots,x(a_n(t)))$, $D_{0+}^{\nu,\mu}$ is the Hilfer fractional derivative, $0\le\nu\le 1, 0<\mu<1$, $-A$ is the infinitesimal generator of an analytic semigroup of bounded linear operators $S(t), t\ge 0$, on a separable Hilbert space H with inner product $\langle.,.\rangle$ and norm $\|.\|$. Let K be another separable Hilbert space with inner

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product $\langle .,.\rangle_K$ and norm $\|.\|_K$. Suppose $\{\omega(t)\}_{t\geq 0}$ is given K-valued Wiener process with a finite trace nuclear covariance operator $Q \geq 0$. We are also employing the same notation $\|.\|$ for the norm $L(K, H)$, where $L(K, H)$ denotes the space of all bounded linear operators from K into H. The functions F, G and g are given functions to be defined later.

2. Preliminaries

In order to derive the existence of mild solutions of Hilfer fractional stochastic integro-differential equations with nonlocal conditions, we need the following basic definitions and Lemmas.

Definition 2.1. (see [20,21]). The Riemann–Liouville fractional integral operator of order $\mu > 0$ for a function f can be defined as

$$I^{\mu} f(t) = \frac{1}{\Gamma(\mu)} \int_0^t \frac{f(s)}{(t-s)^{1-\mu}} ds, \quad t > 0$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2. (see [2]). The Hilfer fractional derivative of order $0 \leq \nu \leq 1$ and $0 < \mu < 1$ is defined as

$$D_{0+}^{\nu,\mu} f(t) = I_{0+}^{\nu(1-\mu)} \frac{d}{dt} I_{0+}^{(1-\nu)(1-\mu)} f(t).$$

Remark.

(i) When $\nu = 0$ and $0 < \mu < 1$, the Hilfer fractional derivative corresponds to the classical Riemann–Liouville fractional derivative: $D_{0+}^{0,\mu} f(t) = I_{0+}^{1-\mu} \frac{d}{dt} f(t) = {}^{L}D_{0+}^{\mu} f(t)$.
(ii) When $\nu = 1$ and $0 < \mu < 1$, the Hilfer fractional derivative corresponds to the classical Caputo fractional derivative: $D_{0+}^{1,\mu} f(t) = I_{0+}^{1-\mu} \frac{d}{dt} f(t) = {}^{C}D_{0+}^{\mu} f(t)$.

Throughout this paper, $(H, \|.\|)$ and $(K, \|.\|_K)$ denote two real Hilbert space.

Let (Ω, Υ, P) be a complete probability space furnished with complete family of right continuous increasing sub σ-algebras $\{\Upsilon_t : t \in J\}$ satisfying $\Upsilon_t \subset \Upsilon$. Let $L_Q(K, H)$ denote the space of all Q-Hilbert-Schmidt operators $\phi : K \to H$. The completion $L_Q(K, H)$ of $L(K, H)$ with respect to the topology induced by the norm $\|.\|_Q$ where $\|\phi\|_Q^2 = \langle \phi, \phi \rangle$ is a Hilbert space with the above norm topology. We suppose that $0 \in \rho(A)$, the resolvent set of A, and the semigroup $S(\cdot)$ is uniformly bounded, that is to say, $\|S(t)\| \leq M$ for some constant $M \geq 1$ and every $t \geq 0$. Then, for $\gamma \in (0, 1]$ it is possible to define the fractional power operator A^{γ}, as a closed linear operator on its domain $D(A^{\gamma})$. Furthermore, the subspace $D(A^{\gamma})$ is dense in H.

We will introduce the following basic properties of A^{γ}.

Theorem 2.1. *(see [22]).*

(1) *Let $0 < \gamma \leq 1$, then $H_{\gamma} := D(A^{\gamma})$ is a Banach space with the norm $\|x\|_{\gamma} = \|A^{\gamma} x\|, \quad x \in H_{\gamma}$.*
(2) *If $0 < \beta < \gamma \leq 1$, then $D(A^{\gamma}) \hookrightarrow D(A^{\beta})$ and the embedding is compact whenever the resolvent operator of A is compact.*
(3) *For every $0 < \gamma \leq 1$, there exists a positive constant C_{γ} such that*

$$\|A^{\gamma} S(t)\| \leq \frac{C_{\gamma}}{t^{\gamma}}, \quad 0 < t \leq b. \tag{2.1}$$

The collection of all strongly-measurable, square-integrable, H-valued random variables, denoted by $L_2(\Omega, H)$, is a Banach space equipped with norm $\|x(\cdot)\|_{L_2(\Omega,H)} = (E\|x(.,\omega)\|^2)^{\frac{1}{2}}$, where the expectation, E is defined by $E(x) = \int_{\Omega} x(\omega) dP$. An important subspace of $L_2(\Omega, H)$ is given by $L_2^0(\Omega, H) = \{x \in L_2(\Omega, H),\ x \text{ is } \Upsilon_0\text{-measurable}\}$.

Let $C(J, L_2(\Omega, H))$ be the Banach space of all continuous maps from J into $L_2(\Omega, H)$ satisfying the condition $\sup_{t\in J} E\|x(t)\|^2 < \infty$.

Define $Y = \{x : t^{(1-\nu)(1-\mu)} x(t) \in C(J, L_2(\Omega, H))\}$, with norm $\|\cdot\|_Y$ defined by

$$\|\cdot\|_Y = (\sup_{t\in J} E\|t^{(1-\nu)(1-\mu)} x(t)\|^2)^{\frac{1}{2}}.$$

Obviously, Y is a Banach space.

For $x \in H$, we define two families of operators $\{S_{\nu,\mu}(t) : t \geq 0\}$ and $\{P_{\mu}(t) : t \geq 0\}$ by

$$S_{\nu,\mu}(t) = I_{0+}^{\nu(1-\mu)} P_{\mu}(t), \quad P_{\mu}(t) = t^{\mu-1} T_{\mu}(t), \quad T_{\mu}(t) = \int_0^{\infty} \mu\theta \Psi_{\mu}(\theta) S(t^{\mu}\theta) d\theta, \tag{2.2}$$

where

$$\Psi_{\mu}(\theta) = \sum_{n=1}^{\infty} \frac{(-\theta)^{n-1}}{(n-1)!\Gamma(1-n\mu)}, \quad 0 < \mu < 1, \quad \theta \in (0, \infty) \tag{2.3}$$

is a function of Wright-type which satisfies the following inequality

$$\int_0^{\infty} \theta^{\Psi} \Psi_{\mu}(\theta) d\theta = \frac{\Gamma(1+\Psi)}{\Gamma(1+\mu\Psi)} \text{ for } \theta \geq 0. \tag{2.4}$$