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Positive solutions of a predator-prey model with cross-diffusion☆

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ABSTRACT

In this paper, we consider the positive solutions for a predator-prey model with cross-diffusion and Holling type II functional response. In particular, the existence of positive solutions can be established by the bifurcation theory. Moreover, the uniqueness and the exact number of positive solutions is studied when the parameter m is large. Furthermore, for large cross-diffusion rate α with the spatial dimension is less than 5, we can derive the corresponding limit systems and also study the asymptotic behavior of positive solutions. Finally, some numerical simulations are presented to supplement the analysis results in one dimension case. This results give us some important information on the structure of positive solutions.

1. Introduction

In this paper, we consider the following predator-prey model with cross-diffusion

$$\begin{cases} -\Delta[(1+\alpha v)u] = u\left(a - u - \dfrac{cv}{1+mu}\right), & x \in \Omega, \\ -\Delta v = v\left(b - v + \dfrac{du}{1+mu}\right), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in R^N with smooth boundary $\partial\Omega$, a, c, d are positive constants; m, α are non-negative constants; b may change sign. In the first equation of (1.1), the nonlinear diffusion term $\alpha\Delta(uv)$ describes a tendency such that prey species keep away from high-density areas of predator species. This nonlinear term $\alpha\Delta(uv)$ is usually called as the cross-diffusion term.

If $\alpha = 0$, then (1.1) is reduced to the predator-prey model with Holling type II function response, which has extensively studied by many scholars, see [1–4,6,8–11]. In [1], the existence of positive solutions is studied by the bifurcation theory. A good understanding of the existence, stability and number of positive solutions is gained when m is large. In particular, various results on the exact number of positive solutions are established in [8]. Furthermore, in [9], Du and Lou showed that when b, c, d and m fall into some range, the positive solutions form a S-shaped smooth curve, and that Hopf bifurcation occurs along this curve.

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For the case of $\alpha > 0$ and $m = 0$, (1.1) becomes a classical predator-prey model with cross-diffusion, which has also received considerable attention, see [13,15–22,25]. The multiple existence of positive solutions is studied in [15]. It turns out that the system admits a branch of positive solutions, which is S or $\supset$ shaped with respect to a bifurcation parameter. In [16], a predator-prey model with nonlinear diffusions is investigated. Especially, when the spatial dimension is less than 5, a universal bound for positive solutions is established, and the bounded continuum of positive solutions is obtained. Furthermore, in [13], Kadota and Kuto showed that the coexistence region narrows as α increases and derived the effect of large α on the bifurcation structure of the positive solutions.

In recent years, Zhou and Kim [26] considered the case $\alpha > 0$ and $m > 0$. In particular, they studied the existence of positive solutions by using the Leray–Schauder degree theory, and established the uniqueness of positive solutions when $N = 1$. Moreover, Wang and Li [24], they treated b as the bifurcation parameter and gave the global bifurcation structure of positive solutions, furthermore, the effect of cross-diffusion rate α or β tends to infinity is considered, and they also gave the global bifurcation structure of the positive solution set of the limiting system.

In this paper, we are mainly concerned with the qualitative properties of positive solutions of (1.1). In particular, we can show that (1.1) has at least a positive solution for $a \in (a^{**}, \infty)$. Moreover, when m is large enough, (1.1) can be regarded as a regular perturbation and a singular perturbation of the limiting problems. Especially, when a is near a^* or around a^{**}, (1.1) has exactly two positive solutions, one asymptotically stable and the other unstable. Furthermore, for large m, if $a \geq a^{**}$ with a is bounded, then (1.1) has a unique positive solution, which is asymptotically stable. However, when α is large and $1 \leq N \leq 5$, (1.1) only can be regarded as a singular perturbation of the limit problem. By the global bifurcation theory, we show that the branch of positive solutions of limiting system bifurcating from $(\theta_a, 0)$ at $\lambda_1(-\frac{d\theta_a}{1+m\theta_a})$ can reach $(0, s^*\phi_1)$ at λ_1, where s^* is given by Lemma 2.2.

Finally, we introduce some notations and basic facts. Let $\lambda_1(p) < \lambda_2(p) \leq \lambda_3(p) \leq \cdots$ be all eigenvalues of the following problem

$$-\Delta u + p(x)u = \lambda u, \quad u|_{\partial\Omega} = 0,$$

where $p \in C^\sigma(\bar{\Omega})$. It is well known that $\lambda_1(p)$ is simple, real and $\lambda_1(p)$ is strictly increasing in the sense that if $p_1 \lneqq p_2$ implies that $\lambda_1(p_1) < \lambda_1(p_2)$. When $p \equiv 0$, we denote $\lambda_1(0)$ by λ_1. Moreover, we denote by ϕ_1 the eigenfunction corresponding to λ_1 with normalization $\|\phi_1\|_\infty = 1$ and positive in Ω.

It is well known that for any $a > \lambda_1$, the problem

$$-\Delta u = u(a - u), \quad u|_{\partial\Omega} = 0 \tag{1.2}$$

has a unique positive solution denoted it by θ_a. It is also known that θ_a is continuously differentiable, strictly increasing in $(\lambda_1, +\infty)$. Moreover, it is non-degenerate and linearly stable.

The rest of this paper is organized as follows. In Section 2, we give some preliminary results which are needed in the later sections. In Section 3, we study the existence and stability of positive solutions. In Section 4, we establish the uniqueness and the exact number of positive solutions for large m. In Section 5, we consider the asymptotic behavior of positive solutions when the cross-diffusion rate is large enough. In Section 6, we present some numerical simulations that supplement the analysis results in one dimension. Finally, we give some discussion in Section 7.

2. Preliminaries

Let $w = (1 + \alpha v)u$. Then (1.1) can be written as

$$\begin{cases} -\Delta w = u\left(a - u - \dfrac{cv}{1+mu}\right), & x \in \Omega, \\ -\Delta v = v\left(b - v + \dfrac{du}{1+mu}\right), & x \in \Omega, \\ w = v = 0, & x \in \partial\Omega. \end{cases} \tag{2.1}$$

First, we derive a priori estimates for non-negative solutions of (2.1).

Lemma 2.1. *Let (w, v) be a non-negative solution of (2.1). Then*

$$0 \leq u(x) \leq w(x) \leq M(a, \alpha) = a\left(1 + \alpha\left(b + \frac{d}{m}\right)\right), \quad 0 \leq v(x) \leq b + \frac{d}{m}. \tag{2.2}$$

Next, we define the set S_0 with a fixed by

$$S_0(a, s) = \left\{(a, s) \in R^2 : \lambda_1\left(-\frac{a}{1+s\phi_1}\right) = 0, \quad a \geq \lambda_1\right\}, \tag{2.3}$$

and the following sets S_1, S_2 and S_3 with b, c and α fixed by

$$S_1(a, b, \alpha) = \left\{(a, b) \in R^2 : \lambda_1\left(-\frac{a}{1+\alpha\theta_b}\right) = 0, \quad b \geq \lambda_1\right\},$$