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The kite graph is determined by its adjacency spectrum

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ABSTRACT

The kite graph is obtained by appending the complete graph with p vertices to a pendant vertex of the path graph with q vertices. In this paper, we prove that the kite graph is determined by its adjacency spectrum for all p and q.

1. Introduction

In this paper, unless indicated otherwise, all graphs are simple and undirected. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G)$. As usual, P_n, C_n and K_n denotes the path, the cycle and the complete graph with n vertices, respectively. The *order of G* is the number of vertices in G and it is denoted by $|G|$. The *degree* of a vertex v is the number of edges incident to v and it is denoted by $d(v)$. We denote the *maximum degree* and the *minimum degree* of a graph G with $\Delta(G)$ and $\delta(G)$, respectively. For a vertex v of G, $N(v)$ denotes the set of vertices in G which are adjacent to v. An edge of G is said to be a *pendant edge* if and only if it contains a vertex which has degree one. A *clique* of G is a complete subgraph of G and the *size of a clique* is the number of vertices in the clique. The largest possible size of a clique of G is called the *clique number* of G and denoted by $w(G)$. If a clique in G has size 3, then it is called a *triangle* of G. A *petal* is added to a graph when we add a pendant edge and then duplicate this edge to form a pendant 2-cycle. A *blossom* B_k consists of k petals $(k \geq 0)$ attached a single vertex; thus B_0 is just the trivial graph. A graph with blossoms (possibly empty) at each vertex is called a $B-graph$. The *line graph* $L(G)$ of a graph G is the graph whose vertices are the edges of G, with two vertices in $L(G)$ adjacent whenever the corresponding edges in G have exactly one vertex in common. Let H be a graph with the vertex set $\{h_1, \dots, h_n\}$ and let $a_1, \dots, a_n$ be non-negative integers. The *generalized line graph* $GL(H; a_1, \dots, a_n)$ is the graph $L(\widehat{H})$, where $\widehat{H}$ is the $B-graph$ $H(a_1, \dots, a_n)$ obtained from H by adding a_i petals at vertex h_i $(i = 1, \dots, n)$.

Let $A(G)$ be the $(0, 1)-$*adjacency matrix of G* and $P_{A(G)}(\lambda) = det(\lambda I - A(G))$ is the *characteristic polynomial of G* where I is the *identity matrix*. Since the matrix $A(G)$ real and symmetric, its eigenvalues, i.e. all roots of $P_{A(G)}(\lambda)$, are all real numbers and called the *adjacency eigenvalues of G*. We use $\lambda_1(G) \geq \lambda_2(G) \geq \dots \geq \lambda_n(G)$ to denote the adjacency eigenvalues of G. These eigenvalues compose the *adjacency spectrum* of G. The largest adjacency eigenvalue of G is known as its *adjacency spectral radius* and denoted by $\rho(G)$. Let $D(G)$ be the diagonal matrix of vertex degrees of G. Then the matrices $\xi(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ are called the *Laplacian matrix* and the *signless Laplacian matrix* of G, respectively. The *Laplacian spectrum* and the *signless Laplacian spectrum* consists of the eigenvalues of $\xi(G)$ and $Q(G)$, respectively. We use $\mu_1(G) \geq \mu_2(G) \geq \dots \geq \mu_n(G)$ to denote the signless Laplacian eigenvalues of G. Two graphs G and H are said to be *cospectral*

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if they share the same spectrum. A graph G is said to be *determined by its spectrum* if there isn't any other non-isomorphic graph that is cospectral with G.

If a tree has exactly one vertex of degree greater than 2, then it is called a *starlike tree*. A starlike tree with maximum degree Δ is denoted by $T(l_1, l_2, \ldots, l_\Delta)$ such that $T(l_1, l_2, \ldots, l_\Delta) - v = P_{l_1} \cup P_{l_2} \cup \ldots \cup P_{l_\Delta}$ where v is the vertex of degree Δ in the starlike tree. The *kite graph* is obtained by appending a complete graph K_p to a pendant vertex of a path P_q and it is denoted by $Kite_p^q$. Actually, $Kite_p^q$ is the line graph of a starlike tree $T(l_1, l_2, \ldots, l_p)$ where $l_1 = l_2 = \ldots = l_{p-1} = 1$ and $l_p = q + 1$. The line graph of any starlike tree is called a *sunlike graph* [8]. Hence kite graph is also a sunlike graph with some conditions. The *double kite graph*, denoted by $DK(p, q)$, is obtained by appending a complete graph with p vertices to both pendant vertices of a path graph with q vertices [20].

Kite graph is already appeared in the literature several times [1,8,14,17–20]. In [1], authors have shown that among connected graphs with order n and clique number w, the minimum value of adjacency spectral radius is attained for a kite graph. Also, they've given a small interval for adjacency spectral radius of kite graph with any fixed clique number. Nath and Paul [17] have shown that among all graphs, kite graph is the unique graph that maximizes the distance spectral radius. Among all connected graphs with a fixed clique number, the first four smallest values of adjacency spectral radius and the first three smallest values of Laplacian spectral radius are obtained in [18,19], respectively. For a given connected graph G, if u denotes the unique positive eigenvector of length 1 which corresponds to the largest positive eigenvalue of $A(G)$, then u is called the *principal eigenvector of* G. The *principal ratio of* G is $\gamma(G) = u_{max}/u_{min}$ such that u_{max} and u_{min} denotes the largest and the smallest components of the vector u. In [23], Cioabă and Gregory have conjectured that $max\{\gamma(G) : |G| = n \geq 3\}$ is obtained at the kite graph. This conjecture is proven by Tait and Tobin [24]. By the help of all of these papers, it can be said that the kite graph plays an important role on the extremum values of the spectral radiuses of many graph matrices.

Also, Stanić [20] has proved that among connected graphs any double kite graph is determined by its adjacency spectrum. By appending a cycle to a pendant vertex of a path, we obtain the well-known lollipop graph. It is already shown that lollipop graph is determined by its adjacency spectrum [2,10]. Similarly, by appending a complete graph to a pendant vertex of a path, we obtain kite graph. $Kite_p^q$ is called as a *short kite graph* when $q = 1$. From [3,5,21], it can be seen that short kite graph is determined by its adjacency spectrum and Laplacian spectrum. Recently, Das and Liu [22] have proved that short kite graph is determined by its signless Laplacian spectrum when $p \neq 4$ and $p > 3$, and also determined by its distance spectrum when $p \geq 3$. In [14], adjacency characteristic polynomial of kite graph is given and it is shown that there are no two non-isomorphic kite graphs have the same adjacency spectrum. Also, in the same paper, it is proved that $Kite_p^2$ is determined by its adjacency spectrum for all p and it is conjectured that $Kite_p^q$ is determined by its adjacency spectrum for all p and q. By the motivation of the all of these results, in this paper we prove that $Kite_p^q$ is determined by its adjacency spectrum for all p and q.

Additionally, the problem : "Which non-regular graphs with least eigenvalue at least -2 are determined by their adjacency spectrum?" is proposed firstly by van Dam and Haemers [4] and partially answered in some recent results [11,13]. Zhou and Bu [13], have proved that the line graph of a starlike tree with maximum degree Δ is determined by its adjacency spectrum when $\Delta \geq 12$. They have left an open problem for $\Delta < 12$ that is "When is the line graph of a starlike tree determined by its adjacency spectrum?". Since the smallest eigenvalue of the kite graph is greater than -2 and the kite graph is actually line graph of a starlike tree, our result in this paper partially answers the open problems in [4,13].

For $p \leq 2$, $p = 3$ and $q = 0$, clearly we obtain path graph, lollipop graph and complete graph, respectively. It is well-known that these graphs are already determined by their adjacency spectrum [4]. Hence, we will consider the case $p \geq 4$ and $q \geq 3$ in this work.

2. Preliminaries

Lemma 2.1 [4,13]**.** *The adjacency eigenvalues of the path graph with n vertices are* $2\cos\frac{\pi j}{n+1}$ *for* $j = 1, \ldots, n$.

Lemma 2.2 [6] (Interlacing Lemma)**.** *If G is a graph on n vertices with eigenvalues* $\lambda_1(G) \geq \ldots \geq \lambda_n(G)$ *and H is an induced subgraph on m vertices with eigenvalues* $\lambda_1(H) \geq \ldots \geq \lambda_m(H)$, *then for* $i = 1, \ldots, m$

$$\lambda_i(G) \geq \lambda_i(H) \geq \lambda_{n-m+i}(G)$$

Lemma 2.3 [13,15]**.** *Let G be a graph with n vertices and m edges, and let L(G) be the line graph of G. Then*

$$P_{A(L(G))}(x) = (x+2)^{m-n} P_{Q(G)}(x+2)$$

For a graph G, the *subdivision graph* of G, denoted by $S(G)$, is the graph obtained from G by inserting a new vertex in each edge of G.

Lemma 2.4 [13,15]**.** *Let G be a graph with n vertices and m edges, and let S(G) be the subdivision graph of G. Then*

$$P_{A(S(G))}(x) = x^{m-n} P_{Q(G)}(x^2)$$

Theorem 2.5 [12]**.** *Each starlike tree with maximum degree 4 is determined by its signless Laplacian spectrum.*

Lemma 2.6 [15]**.** *The starlike tree* $T(l_1, \ldots, l_\Delta)$ *with* $\Delta \geq 5$ *is determined by its signless Laplacian spectrum.*