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Hyers–Ulam stability of first-order nonhomogeneous linear difference equations with a constant stepsize

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ABSTRACT

The present paper deals with Hyers–Ulam stability of the first-order linear difference equation $\Delta_h x(t) - ax(t) = f(t)$ on $h\mathbb{Z}$, where $\Delta_h x(t) = (x(t+h) - x(t))/h$ and $h\mathbb{Z} = \{hk | \, k \in \mathbb{Z}\}$ for the constant stepsize $h > 0$; a is a real number; $f(t)$ is a real-valued function on $h\mathbb{Z}$. The main purpose of this paper is to find the best HUS constant on $h\mathbb{Z}$. Several relationships between solutions of two different perturbed difference equations are also given.

1. Introduction

For given $h > 0$ we define

$$\Delta_h x(t) = \frac{x(t+h) - x(t)}{h} \quad \text{and} \quad h\mathbb{Z} = \{hk | \, k \in \mathbb{Z}\}.$$

We call such h the "*stepsize*". In this paper, we consider the first-order nonhomogeneous linear difference equation

$$\Delta_h x(t) - ax(t) = f(t) \tag{1.1}$$

on $h\mathbb{Z}$, where a is a real number, and $f(t)$ is a real-valued function on $h\mathbb{Z}$. It is well-known that the global existence and uniqueness of solutions of (1.1) are guaranteed for the initial-value problem.

Let I be a nonempty open interval of $\mathbb{R}$. We define $T = h\mathbb{Z} \cap I$ and

$$T^* = \begin{cases} T \setminus \{\max T\}, & \text{if the maximum of } T \text{ exists,} \\ T, & \text{otherwise.} \end{cases}$$

Noticing that if a function $\phi(t)$ exists on T, then $\Delta_h \phi(t)$ exists on T^*. We now describe the definition of the stability. We say that (1.1) has "*Hyers–Ulam stability*" on T if there exists a constant $K > 0$ with the following property: Let $\varepsilon > 0$ be a given arbitrary constant. If a function $\phi : T \to \mathbb{R}$ satisfies $|\Delta_h \phi(t) - a\phi(t) - f(t)| \le \varepsilon$ for all $t \in T^*$, then there exists a solution $x : T \to \mathbb{R}$ of (1.1) such that $|\phi(t) - x(t)| \le K\varepsilon$ for all $t \in T$. We call such K a "*HUS constant*" for (1.1) on T. In addition, we call the minimum of HUS constants for (1.1) on T the "*best HUS constant*". Hyers–Ulam stability is originated from a stability problem in the field of functional equations (see [1,4–6,11,14,23,27,28]). In Sections 1, 2 and 4, we treat only the case

$$a \neq 0, \; -\frac{1}{h} \; \text{and} \; -\frac{2}{h},$$

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because of the following reasons.

Remark 1.1. If $a=-1/h$, then we no longer have a first order difference equation. It is known that if $f(t)\equiv 0$ with $a=0$ or $a=-2/h$, then (1.1) does not have Hyers–Ulam stability on $h\mathbb{Z}$ (see [17]).

As it is well-known that (1.1) is an approximation of the ordinary differential equation $x'-ax=f(t)$, since $x'(t)$ can be approximated by $\Delta_h x(t)$. Hyers–Ulam stability of ordinary differential equations, difference equations and linear operators has been studied by many researchers (see [2,3,7–10,12,13,15–22,24–26,29,30]).

In 2017, the author [17] discussed the problem 'how does the stepsize influence the best HUS constant for the homogeneous linear equation

$$\Delta_h x(t)-ax(t)=0 \tag{1.2}$$

on $h\mathbb{Z}$?' In the small stepsize case, the best HUS constant of difference equation is the same as that of ODE. On the other hand, in the large stepsize case, the best HUS constant of difference equation is different from that of ODE. In order to derive these facts, the following theorems played an important role.

Theorem A. *Let $\varepsilon>0$ be a given arbitrary constant. Suppose that a function $\phi: T\to\mathbb{R}$ satisfies $|\Delta_h\phi(t)-a\phi(t)|\le\varepsilon$ for all $t\in T^*$, where $a\neq 0$ and $a>-1/h$. Then one of the following holds:*

(i) *if $a>0$ and $\max T$ exists, then any solution $x(t)$ of (1.2) with $|\phi(\tau)-x(\tau)|<\varepsilon/a$ satisfies that $|\phi(t)-x(t)|<\varepsilon/a$ for all $t\in T$, where $\tau=\max T$;*
(ii) *if $a>0$ and $\max T$ does not exist, then $\lim_{t\to\infty}\phi(t)(ah+1)^{-t/h}$ exists, and there exists exactly one solution*

$$x(t)=\left\{\lim_{t\to\infty}\phi(t)(ah+1)^{-\frac{t}{h}}\right\}(ah+1)^{\frac{t}{h}}$$

of (1.2) such that $|\phi(t)-x(t)|\le\varepsilon/a$ for all $t\in T$;
(iii) *if $-1/h<a<0$ and $\min T$ exists, then any solution $x(t)$ of (1.2) with $|\phi(\sigma)-x(\sigma)|<\varepsilon/|a|$ satisfies that $|\phi(t)-x(t)|<\varepsilon/|a|$ for all $t\in T$, where $\sigma=\min T$;*
(iv) *if $-1/h<a<0$ and $\min T$ does not exist, then $\lim_{t\to-\infty}\phi(t)(ah+1)^{-t/h}$ exists, and there exists exactly one solution*

$$x(t)=\left\{\lim_{t\to-\infty}\phi(t)(ah+1)^{-\frac{t}{h}}\right\}(ah+1)^{\frac{t}{h}}$$

of (1.2) such that $|\phi(t)-x(t)|\le\varepsilon/|a|$ for all $t\in T$.

Theorem B. *Let $\varepsilon>0$ be a given arbitrary constant. Suppose that a function $\phi: T\to\mathbb{R}$ satisfies $|\Delta_h\phi(t)-a\phi(t)|\le\varepsilon$ for all $t\in T^*$, where $a\neq -2/h$ and $a<-1/h$. Then one of the following holds:*

(i) *if $-2/h<a<-1/h$ and $\min T$ exists, then any solution $x(t)$ of (1.2) with $|\phi(\sigma)-x(\sigma)|<\varepsilon/(a+2/h)$ satisfies that $|\phi(t)-x(t)|<\varepsilon/(a+2/h)$ for all $t\in T$, where $\sigma=\min T$;*
(ii) *if $-2/h<a<-1/h$ and $\min T$ does not exist, then $\lim_{t\to-\infty}\phi(t)(ah+1)^{-t/h}$ exists, and there exists exactly one solution*

$$x(t)=\left\{\lim_{t\to-\infty}\phi(t)(ah+1)^{-\frac{t}{h}}\right\}(ah+1)^{\frac{t}{h}}$$

of (1.2) such that $|\phi(t)-x(t)|\le\varepsilon/(a+2/h)$ for all $t\in T$;
(iii) *if $a<-2/h$ and $\max T$ exists, then any solution $x(t)$ of (1.2) with $|\phi(\tau)-x(\tau)|<\varepsilon/|a+2/h|$ satisfies that $|\phi(t)-x(t)|<\varepsilon/|a+2/h|$ for all $t\in T$, where $\tau=\max T$;*
(iv) *if $a<-2/h$ and $\max T$ does not exist, then $\lim_{t\to\infty}\phi(t)(ah+1)^{-t/h}$ exists, and there exists exactly one solution*

$$x(t)=\left\{\lim_{t\to\infty}\phi(t)(ah+1)^{-\frac{t}{h}}\right\}(ah+1)^{\frac{t}{h}}$$

of (1.2) such that $|\phi(t)-x(t)|\le\varepsilon/|a+2/h|$ for all $t\in T$.

Remark 1.2. The assertions (iii) and (iv) in Theorem A express the results when the stepsize is sufficiently small. On the other hand, Theorem B expresses the result when the stepsize is large. Furthermore, the assertions (ii) and (iv) in Theorems A and B imply that the best HUS constant for (1.2) on $h\mathbb{Z}$ is

$$B(a,h)=\begin{cases}\dfrac{1}{|a|}, & \text{if}\quad a>0\ \text{ or }\ 0<h<-\dfrac{1}{a},\\[2ex] \dfrac{1}{|a+2/h|}, & \text{if}\quad -\dfrac{1}{a}<h<-\dfrac{2}{a}\ \text{ or }\ -\dfrac{2}{a}<h\end{cases}$$

(see [17, Remarks 4.4 and 4.6]). This constant is rewritten as

$$B(a,h)=\frac{1}{||a+1/h|-1/h|}. \tag{1.3}$$

The purpose of this paper is to find the best HUS constant for (1.1) on $h\mathbb{Z}$, by using the obtained new theorems which will be presented in Section 2. In addition, we will also find an explicit solution $x(t)$ of (1.1) such that $|\phi(t)-x(t)|$ is less than