Applied Mathematics and Computation 331 (2018) 287–296

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Optimizing Hamiltonian panconnectedness for the crossed cube architecture

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ARTICLE INFO

Keywords:
Hamiltonian
Panconnected
Interconnection network
Crossed cube
Path embedding

ABSTRACT

A graph G of k vertices is panconnected if for any two distinct vertices x and y, it has a path of length l joining x and y for any integer l satisfying $d_G(x, y) \leq l \leq k-1$, where $d_G(x, y)$ denotes the distance between x and y in G. In particular, when $k \geq 3$, G is called Hamiltonian r-panconnected if for any three distinct vertices x, y, and z, there exists a Hamiltonian path P of G with $d_P(x, y) = l$ such that $P(1) = x$, $P(l+1) = y$, and $P(k) = z$ for any integer l satisfying $r \leq l \leq k-r-1$, where $P(i)$ denotes the ith vertex of path P for $1 \leq i \leq k$. Then, this paper shows that the n-dimensional crossed cube, which is a popular variant of the hypercube topology, is Hamiltonian $(\lceil \frac{n+1}{2} \rceil + 1)$-panconnected for $n \geq 4$. The lower bound $\lceil \frac{n+1}{2} \rceil + 1$ on the path length is sharp, which is the shortest that can be embedded between any two distinct vertices with dilation 1 in the n-dimensional crossed cube.

1. Introduction

Various types of interconnection networks have been proposed for parallel and distributed computing [15,24,27]. A typical methodology to model the topological structure of an interconnection network is based on the graph theory. Widely applied graph structures include trees, paths, cycles, etc. Among the many kinds of network topologies, the hypercube [26] is a popular architecture, and the crossed cube [11] is a well-known one of hypercube variants [1,9,10,29]. Compared with the hypercube, the crossed cube has a smaller diameter but relies on a more complicated shortest-path routing [4,11]. Additionally, the crossed cube has many other attractive properties [4–6,12–14,16,18,21,28]. Before proceeding to study topological properties of the crossed cube, we introduce some fundamental graph-theory definitions and notations that will be used in the paper.

Throughout this paper, graphs are finite, simple, and undirected. For those not defined here, we follow the standard terminology given by Bondy and Murty [3]. An *undirected graph* G consists of a vertex set $V(G)$ and an edge set $E(G)$, where $V(G)$ is a finite set, and $E(G)$ is a subset of all unordered pairs of distinct elements in V. For notational simplicity, uv denotes the edge joining vertices u and v. Equivalently, u and v are *adjacent* to each other in G. A graph H is a *subgraph* of G, denoted by $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, and H is a *spanning subgraph* of G (or equivalently, H *spans* G) if $V(H) = V(G)$. Let S be a nonempty subset of $V(G)$. The subgraph of G *induced* by S is a graph whose vertex set is S and whose edge set consists of all the edges of G joining any two vertices in S.

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https://doi.org/10.1016/j.amc.2018.03.002
0096-3003/

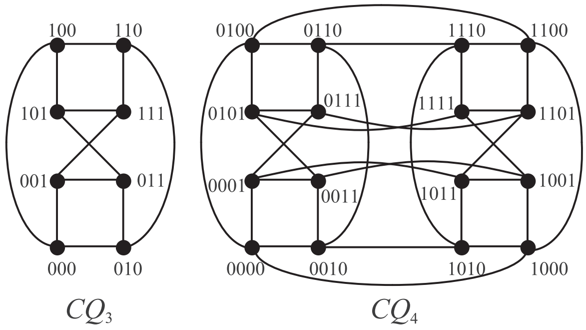

Fig. A.1. Illustrating CQ_3 and CQ_4.

A *path* P of length l, $l \geq 1$, from vertex u to vertex v in a graph G is defined as an ordered sequence of distinct vertices $x_1\ x_2\ \cdots\ x_{l+1}$ such that $x_1 = u$, $x_{l+1} = v$, and $x_i x_{i+1} \in E(G)$ for every i, $1 \leq i \leq l$. For convenience, we write P as $x_1\ \cdots\ x_i\ R\ x_j\ \cdots\ x_{l+1}$ if $R \subseteq P$ and $R = x_i \cdots x_j$ for $1 \leq i < j \leq l+1$. We also use uPv to stand for a path P with u and v as the starting and ending vertices, respectively. The length of P is denoted by $\ell(P)$. The ith vertex of P is denoted by $P(i)$; i.e., $P(i) = x_i$. In particular, let $rev(P)$ represent the reverse of P; i.e., $rev(P) = x_{l+1}\ x_l\ \cdots\ x_1$. The *distance* between two distinct vertices u and v in G, denoted by $d_G(u, v)$, is the length of the shortest path connecting u and v. A *cycle* is a path with at least three vertices such that the first vertex is linked to the last one. For clarity, a cycle of length l, $l \geq 3$, is represented as $x_1\ x_2\ \cdots\ x_l\ x_1$ and is simply denoted by C_l. A *Hamiltonian path* (respectively, *Hamiltonian cycle*) of G is a path (respectively, cycle) that spans G. Then, a graph G is *Hamiltonian* if it contains at least one Hamiltonian cycle, and G is *Hamiltonian connected* if there exists a Hamiltonian path of G joining each pair of distinct vertices.

A graph G of k vertices is *panconnected* [2] if for any two distinct vertices x and y, it has a path of length l joining x and y for any integer l satisfying $d_G(x, y) \leq l \leq k-1$. Strictly speaking, a panconnected graph must contain a path of length 2 joining any two adjacent vertices. However, there does not exist any C_3 in many well-known interconnection networks, including hypercubes, crossed cubes, star graphs, and so on. Hence, we say that a graph G of k vertices is *r-panconnected* if for any two distinct vertices x and y, it has a path of length l joining x and y for $r \leq l \leq k-1$. This property has been widely addressed for embedding paths on many kinds of interconnection networks, such as the hypercube, the crossed cube, and the augmented cube [5,13,14,19,20,25]. For example, Fan et al. [14] proved that the n-dimensional crossed cube is $(\lceil \frac{n+1}{2} \rceil + 1)$-panconnected. Recently, Kung and Chen [17] extended the definition of r-panconnectedness to propose a new property: a graph G of k vertices, $k \geq 3$, is *Hamiltonian r-panconnected* if for any three distinct vertices x, y, and z, there exists a Hamiltonian path P of G with $d_P(x, y) = l$ such that $P(1) = x$, $P(l+1) = y$, and $P(k) = z$ for any integer l satisfying $r \leq l \leq k-r-1$. In this paper, we aim to show that the n-dimensional crossed cube is Hamiltonian $(\lceil \frac{n+1}{2} \rceil + 1)$-panconnected for $n \geq 4$. The lower bound $(\lceil \frac{n+1}{2} \rceil + 1)$ on the path length is sharp, which is the smallest that can be achieved between two arbitrary vertices with dilation 1.

The remainder of this paper is structured as follows. In Section 2, we introduce the definition and topological properties of the crossed cube. The main theorem and its proof are described in Section 3. Finally, some concluding remarks are given in Section 4.

2. The crossed cube and its properties

The n-dimensional crossed cube, denoted by CQ_n, has 2^n vertices, each of which corresponds to an n-bit binary string. To define the crossed cube, we first introduce an additional concept, “pair-related”.

Definition 1 [11]**.** Two 2-bit binary strings $\mathbf{x} = x_2 x_1$ and $\mathbf{y} = y_2 y_1$ are pair-related, denoted by $\mathbf{x} \sim \mathbf{y}$, if and only if $(\mathbf{x}, \mathbf{y}) \in \{(00, 00), (10, 10), (01, 11), (11, 01)\}$.

The formal definition of CQ_n is given below, and Fig. A.1 depicts CQ_3 and CQ_4.

Definition 2 [11]**.** The n-dimensional crossed cube CQ_n is recursively constructed as follows:

(i) CQ_1 is a complete graph with vertex set $\{0, 1\}$.
(ii) CQ_2 is isomorphic to a C_4 with vertex set $\{\mathbf{u} = 00, \mathbf{v} = 01, \mathbf{x} = 10, \mathbf{y} = 11\}$ and edge set $\{\mathbf{uv}, \mathbf{ux}, \mathbf{vy}, \mathbf{xy}\}$.
(iii) For $n \geq 3$, let CQ^0_{n-1} and CQ^1_{n-1} denote two copies of CQ_{n-1} with $V(CQ^0_{n-1}) = \{0u_{n-1}u_{n-2}\ldots u_1 \mid u_i = 0 \text{ or } 1 \text{ for } 1 \leq i \leq n-1\}$ and $V(CQ^1_{n-1}) = \{1u_{n-1}u_{n-2}\ldots u_1 \mid u_i = 0 \text{ or } 1 \text{ for } 1 \leq i \leq n-1\}$. Then, CQ_n is formed by connecting CQ^0_{n-1} and CQ^1_{n-1} with 2^{n-1} edges so that a vertex $\mathbf{u} = 0u_{n-1}u_{n-2}\ldots u_1$ of CQ^0_{n-1} is adjacent to a vertex $\mathbf{v} = 1v_{n-1}v_{n-2}\ldots v_1$ of CQ^1_{n-1} if and only if (1) $u_{n-1} = v_{n-1}$ if n is even, and (2) $u_{2i}u_{2i-1} \sim v_{2i}v_{2i-1}$ for all i, $1 \leq i \leq \lfloor \frac{n-1}{2} \rfloor$.

For the sake of convenience, the recursive construction of CQ_n is denoted by $CQ_n \cong CQ^0_{n-1} \otimes CQ^1_{n-1}$. In general, let $CQ^{\mathbf{x}}_{n-|\mathbf{x}|}$ be the subgraph of CQ_n induced by the set of vertices with the prefix $\mathbf{x}$, where $\mathbf{x}$ is any k-bit binary string with $|\mathbf{x}| = k < n$.