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Existence result for differential variational inequality with relaxing the convexity condition☆

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ABSTRACT

In this paper, a class of differential variational inequalities are studied, and a new approach is introduced to relax the convexity condition. Firstly, an existence theorem of Carathéodory weak solution for the differential variational inequalities is established. Secondly, an algorithm for solving the problem is developed and the convergence analysis for the algorithm is given. Finally, a numerical example is reported to illustrate the proposed algorithm.

1. Introduction

In 2008, Pang and Stewart [14] introduce and study a class of differential variational inequalities. It is a powerful modeling paradigm, which unifies several mathematical problems that include differential inclusions, dynamic complementarity systems, differential equilibrium problems, and so on. Since differential variational inequality has a wide range of applications, particularly in finance, economics, transportation, and engineering sciences, many researchers have been attracted to its development, see, for example, [1,3,5–9,17–20].

In previous studies, the mathematical model of differential variational inequality is often in the form of a differential inclusion. By employing the differential inclusion theorem (Lemma 6.1 in [14]), the model admits a solution and then the measurability of the solution can be obtained by a version of Filippov's implicit function theorem (Lemma 6.3 in [14]). In this way, the value of the mapping needs to be convex. It means that the solution set of static variational inequality should be convex. However, there are many cases that the solution set is not convex, see, for example, [10–13,16]. Therefore, a new approach is introduced to relax the convexity condition in this paper. The remainder of this paper is organized as follows. In Section 2, we present some preliminaries. In Section 3, we consider the existence of Carathéodory weak solution of differential variational inequality. In Section 4, an algorithm for solving differential variational inequality is developed, and the convergence analysis for the algorithm is given. In Section 5, a numerical example is reported to illustrate the proposed algorithm.

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2. Preliminaries

In this paper, we study the following differential variational inequality (DVI, for short),

$$\begin{cases} \langle G(z(t)) + F(u(t)), v - u(t)\rangle \geq 0, \quad \forall v \in K, \\ \dot{z}(t) = Cz(t) + Du(t), \\ z(0) = z^0, \end{cases} \tag{2.1}$$

where $\dot{z}(t) = \dfrac{dz(t)}{dt}$, $t \in [0, T]$, $K \subset R^m$, G: $R^n \to R^m$, F: $R^m \to R^m$, C and D are $n \times n$ matrix and $n \times m$ matrix, respectively.

Let $|\cdot|$ denote the Euclidean norm, and

$$\|f(\cdot,\cdot)\|_{\mathcal{C}} := \sup_{(z,u)\in R^n\times K} |f(z,u)|, \tag{2.2}$$

$$\|u(\cdot)\|_{L^2} := \left(\int_0^T |u(t)|^2 dt\right)^{\frac{1}{2}}, \tag{2.3}$$

$L^2([0, T], K)$ denotes the set of all the measurable functions u: $[0, T] \to K$, that satisfying $\|u(\cdot)\|_{L^2} < \infty$, and $\mathcal{C}([0,T], R^n)$ denotes the set of all the continuous functions u: $[0, T] \to R^n$.

Definition 2.1. A mapping F: $L^2([0, T], K) \to L^2([0, T], K)$ is said to be strongly continuous iff, for any $\{u_n(\cdot)\} \subset L^2([0, T], K)$ with $u_n(\cdot)$ weakly convergent to $u(\cdot)$ by the norm $\|\cdot\|_{L^2}$, we have $F(u_n(\cdot))$ convergent to $F(u(\cdot))$ by the norm $\|\cdot\|_{L^2}$.

Let $\mathrm{SOL}(K, G(z(t)) + F(\cdot))$ denote the solution set of the following variational inequality: finding $u(t) \in K$ such that,

$$\langle G(z(t)) + F(u(t)), v - u(t)\rangle \geq 0, \quad \forall v \in K.$$

A pair $(z(\cdot), u(\cdot))$ is called a Carathéodory weak solution of problem (2.1) iff $z(\cdot)$ is an absolutely continuous function on $[0, T]$ and satisfies the following differential equation for almost all $t \in [0, T]$:

$$\dot{z}(t) = f(z(t), u(t)), \quad z(0) = z^0,$$

and $u(\cdot) \in L^2([0, T], K)$ and $u(t) \in \mathrm{SOL}(K, G(z(t)) + F(\cdot))$ for almost all $t \in [0, T]$. We denote the set of Carathéodory weak solutions for DVI (2.1) by SOL(DVI(2.1)).

In addition, we shall recall some preliminary lemmas.

Lemma 2.1 [4]**.** *Let K be a compact and convex subset of R^m, F: $K \to R^m$ be a continuous mapping. Then, there exists $u \in K$ such that*

$$\langle F(u), v - u\rangle \geq 0, \quad \forall v \in K.$$

The following lemma can be derived from the proposition in [14] (see, page 349).

Lemma 2.2. *Let f be continuous on $R^n \times R^m$, G be continuous on R^n, F be continuous on R^m. Then $(z(\cdot), u(\cdot)) \in$ SOL(DVI(2.1)) if and only if the pair $(z(\cdot), u(\cdot))$ satisfies the following three relations:*

(i) *for any $t \in [0, T]$,*

$$z(t) = z^0 + \int_0^t f(z(\tau), u(\tau))d\tau,$$

(ii) *for all $v \in L^2([0,T], K)$,*

$$\int_0^T \langle G(z(\tau)) + F(u(\tau)), v(\tau) - u(\tau)\rangle d\tau \geq 0,$$

(iii) *the initial condition $z(0) = z^0$.*

The following lemma is Schauder fixed point theorem.

Lemma 2.3 [15]**.** *Let A be a closed and convex set in a Banach space, F: $A \to A$ be continuous, and the closure of $F(A)$ be compact. Then there exists a fixed point $z \in A$ such that $F(z) = z$.*

Lemma 2.4 [2]**.** *A function g: $R^n \to R^m$ is measurable iff there is a sequence $\{g_n\}$ of finite, real-valued, step functions such that $g_n \to g$ pointwise.*