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The comparative index and transformations of linear Hamiltonian differential systems

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ABSTRACT

In this paper we investigate mutual oscillatory behaviour of two linear differential Hamiltonian systems related via symplectic transformations. The main result extends our previous results in [30], where we presented new explicit relations connecting the multiplicities of proper focal points of conjoined bases $Y(t)$ of the Hamiltonian system and the transformed conjoined bases $\tilde{Y}(t) = R^{-1}(t)Y(t)$. In the present paper we omit restrictions on the symplectic transformation matrix $R(t)$ concerning the constant rank of its components. As consequences of the main result we prove generalized reciprocity principles which formulate new sufficient conditions for $R(t)$ concerning preservation of (non)oscillation of the abnormal Hamiltonian systems as $t \to \infty$. The main tool of the paper is the comparative index theory for discrete symplectic systems implemented into the continuous case.

1. Introduction

In this paper we investigate oscillation properties of solutions of the linear Hamiltonian systems [1]

$$y'(t) = J\mathcal{H}(t)y(t),\ \mathcal{H}(t) = \begin{bmatrix} -C(t) & A^T(t) \\ A(t) & B(t) \end{bmatrix},\ \mathcal{H}(t) = \mathcal{H}^T(t),\ J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix},\ t \in [a, \infty) \tag{1.1}$$

and

$$\tilde{y}'(t) = J\tilde{\mathcal{H}}(t)\tilde{y}(t),\ \tilde{\mathcal{H}}(t) = \begin{bmatrix} -\tilde{C}(t) & \tilde{A}^T(t) \\ \tilde{A}(t) & \tilde{B}(t) \end{bmatrix},\ \tilde{\mathcal{H}}(t) = \tilde{\mathcal{H}}^T(t),\ t \in [a, \infty) \tag{1.2}$$

which are related via the symplectic transformation

$$\tilde{y}(t) = R^{-1}(t)y(t), \quad R(t) = \begin{bmatrix} L(t) & M(t) \\ K(t) & N(t) \end{bmatrix}, \quad R^T(t)JR(t) = J,\ t \in [a, \infty). \tag{1.3}$$

Here we assume that the $n \times n$ blocks of $\mathcal{H}(t)$ are real piecewise continuous matrix functions of t and that $R(t) \in C_p^1$ (i.e., $R(t)$ is continuous with piecewise continuous $R'(t)$). It is well known [2] that if $R(t) \in \mathbb{R}^{2n \times 2n}$ is symplectic, then (1.2) is also a linear Hamiltonian system, in more details,

$$\tilde{\mathcal{H}}(t) = R^T(t)(\mathcal{H}(t) - J^T R'(t)R^{-1}(t))R(t) = \tilde{\mathcal{H}}^T(t), \tag{1.4}$$

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where $J^T R'(t)R^{-1}(t) = JR'(t)JR^T(t)J$ is symmetric by (1.3). Throughout the paper we assume that systems (1.1), (1.2) obey the so-called Legendre conditions

$$B(t) \geq 0,\ t \in t \in [a, \infty), \tag{1.5}$$

$$\tilde{B}(t) \geq 0,\ t \in t \in [a, \infty), \tag{1.6}$$

where $A \geq 0$ means that the symmetric matrix A is nonnegative definite.

The transformation theory of (1.1) is a classical research topic in the oscillation theory which is highly important for theoretical and practical applications (see, for example, [2–12] and references therein). In this paper, we refine the classical results applying a novel tool which we call the *comparative index theory* (see [13,14]). The comparative index was introduced and elaborated for applications in the discrete oscillation theory for the symplectic systems [15]

$$Y_{k+1} = \mathcal{S}_k Y_k, \quad \mathcal{S}_k^T J \mathcal{S}_k = J \tag{1.7}$$

which are discrete counterparts of (1.1).

The natural problem of the oscillation theory is to look for invariants of transformation (1.3), in particular, to ask when this transformation preserves oscillatory properties of transformed systems. In the classical transformation theory, such as in [2,4,6,8,9] systems (1.1), (1.2) are traditionally studied under controllability (or normality) assumption, see [16, Section 4.1]. Controllability means that the solutions $y(t) = \binom{x(t)}{u(t)}$ of (1.1) are not degenerate in the first component, that is, whenever $x(t) = 0$ on a subinterval of $[a, b]$, then also $u(t) = 0$ in this subinterval. By [16, Theorem 4.1.3], conditions (1.5) and (1.6) the controllability assumption yield that the focal points of $2n \times n$ matrix solutions $Y(t) = \binom{X(t)}{U(t)}$, $\tilde{Y}(t) = \binom{\tilde{X}(t)}{\tilde{U}(t)}$ of (1.1), (1.2) such that

$$X^T(t)U(t) = U^T(t)X(t), \quad \operatorname{rank} Y(t) = n, \quad \tilde{X}^T(t)\tilde{U}(t) = \tilde{U}^T(t)\tilde{X}(t), \quad \operatorname{rank} \tilde{Y}(t) = n \tag{1.8}$$

(the so-called *conjoined bases*) are isolated. The multiplicity of such focal points at t_0 is then the dimension of the kernels of $X(t_0)$, $\tilde{X}(t_0)$, i.e., $m(t_0) := \operatorname{def} X(t_0)$, $\tilde{m}(t_0) := \operatorname{def} \tilde{X}(t_0)$. Using the controllability assumption and (1.5), (1.6) it has been shown in [2,6] that for $R(t) = J$ system (1.1) is nonoscillatory if and only if the reciprocal system (1.2) is nonoscillatory. This statement is now commonly referred as *reciprocity principle* for Hamiltonian systems since it extends the well-known fact that under the assumption $r(t)$, $p(t) > 0$ the second order equation $(r(t)x')' + p(t)x = 0$ is oscillatory if and only if the so-called reciprocal equation $(\frac{1}{p(t)}u')' + \frac{1}{r(t)}u = 0$ is oscillatory. In [8] the reciprocity-type statement was extended under natural additional assumptions to general transformation (1.3). In particular, it is proved in [8, Theorem 1], that under the assumptions (1.5), (1.6), and $\det M(t) \neq 0$, $t \in [a, \infty)$ for the transformation matrix $R(t)$ in (1.3) the controllable systems (1.1), (1.2) oscillate or do not oscillate simultaneously. The similar statement also holds under the assumptions (1.5), (1.6), and $\det K(t) \neq 0$, $C(t) \leq 0$, $\tilde{C}(t) \leq 0$, $t \in [a, \infty)$ (see [8, Theorem 2]).

In [17,18] the concept of *proper focal points* of conjoined bases for possibly abnormal linear Hamiltonian systems was introduced. First Sturmian-type results concerning the multiplicities of proper focal points for differential Hamiltonian systems without normality and their applications are presented in [19–22]. Based on the comparative index approach these results were further developed in [23,24].

Main results in this paper can be regarded as continuous analogs of the results of the discrete time transformation theory for (1.7). First results of this theory were presented in [15,25,26] and then generalized using the comparative index notion in [27–29]. In [30] we presented new explicit relations between the multiplicities of proper focal points of conjoined bases of (1.1), (1.2) under the restriction

$$\operatorname{rank} M(t) = \text{const},\ t \in [a, \infty). \tag{1.9}$$

In the present paper we extend the results in [30] replacing (1.9) by the assumption that rank $M(t)$ is a piecewise constant function of $t \in [a, \infty)$.

The paper is organized as follows. In the next section, we recall some basic concepts of the oscillation theory for (1.1) and properties of the comparative index. We prove one of the most important results of this paper, the so-called local transformation theorem (see Theorem 2.5) connecting the multiplicities of focal points of conjoined bases of (1.1) and (1.2) in a small neighborhood of the fixed t_0 for the case when (1.9) does not hold. We show that in general case the mutual oscillatory behaviour of two systems (1.1), (1.2) depends on the local oscillation properties of the principal solutions of (1.1), (1.2) at t_0 and $R(t)$, where t_0 are points of the discontinuity of rank $M(t)$.

Section 3 is devoted to main consequences of Theorem 2.5. In particular, we prove the global transformation theorem (see Theorem 3.3) presenting new explicit relations between the multiplicities of proper focal points of conjoined bases of (1.1), (1.2) in $(a, b]$, $b < \infty$ and derive the generalized reciprocity principle (Theorem 3.5) for the case when (1.9) does not hold. We also provide examples which illustrate Theorems 3.3, 3.5.

The results of this paper would be incomplete without mentioning an approach based on the local symplectic factorization of the transformation matrix $R(t)$

$$R(t) = R_1(t)R_2(t)\dots R_p(t),\ t \in (t_0 - \delta, t_0 + \delta), \tag{1.10}$$