Applied Mathematics and Computation 330 (2018) 201–212

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Almost sure exponential stability of implicit numerical solution for stochastic functional differential equation with extended polynomial growth condition

Linna Liu [a], Feiqi Deng [a,*], Ting Hou [b]

[a] School of Automation Science and Engineering, South China University of Technology, Guangzhou 510640, China
[b] College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, China

A R T I C L E I N F O

Keywords:
Stochastic functional differential equation
Almost sure exponential stability
Polynomial growth condition
Semi-martingale convergence theorem
Backward Euler–Maruyama method

A B S T R A C T

Stability of numerical solutions to stochastic differential equations (SDEs) has received an increasing attention from researchers in applied mathematics and engineering areas, but there has been so far little work on the stability analysis of numerical solutions to highly nonlinear stochastic functional differential equations (SFDEs). The aim of this paper is to study the almost sure exponential stability of the backward Euler scheme for highly nonlinear SFDEs. Firstly, a stability criterion is established for the almost sure exponential stability of the underlying nonlinear SFDE under an extended polynomial growth condition, the global existence of the solutions of the underlying equation is simultaneously shown with a new technique, the Fatou's Lemma. Secondly, the almost sure exponential stability of the backward Euler scheme is investigated by contrast. The stability criterion shows that the numerical scheme preserves the almost sure exponential stability of the analytic solution under the same growth condition, which improves some related result in literature as a corollary by way. To describe the scheme conveniently, a concept, the interpolation segment, is formally proposed in the paper, which provides a way for constructing a discretized approximation to a continuous function applied in the scheme. To achieve the required results, the classical nonnegative semi-martingale convergence theorem is reformed to a practical version, namely the so called practical nonnegative semi-martingale boundedness lemma in the paper. At the end of the paper, a numerical example is proposed to illustrate the adaption of the growth condition assumed in the paper.

1. Introduction

Stochastic modeling plays an important role in describing the dynamical systems in many branches of science and industries. Since the introduction of Itô formula and related stochastic analysis tools, SDEs have been used successfully to model such systems in many application fields as those in biology, epidemiology, mechanics, control theory and so on. Combined with the time delay issue, the stability of SFDEs, including stochastic delay differential equations (SDDEs), has received lots of attentions over past twenty years and many important results have been established by many researchers (see e.g. [1–5] etc). Generally, SFDEs admit no explicit solution. Therefore, it seems to be interesting and necessary to study the numerical solutions of SFDEs. However, so far, little is known about the stability of numerical solutions about SFDEs as

* Corresponding author.
E-mail address: aufqdeng@scut.edu.cn (F. Deng).

https://doi.org/10.1016/j.amc.2018.02.031
0096-3003/

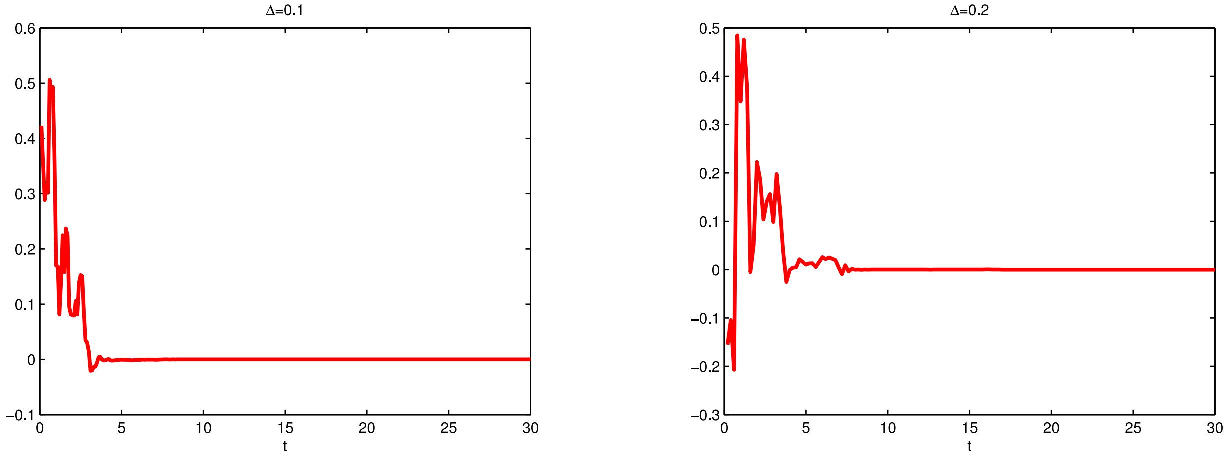

Fig. 1. Backward Euler–Maruyama numerical simulation.

yet, not to mention the highly nonlinear SFDEs due to the technical difficulties for analysis. Recently, Milošević [6] studied the almost sure asymptotically exponential stability of the Euler–Maruyama solution (EM) to some highly nonlinear neutral stochastic differential equations with time-dependent delay under the Khasminskii-type conditions. Wu et al. [7] shew that EM methods can reproduce the moment exponential stability of the exact solution to SFDEs under the linear growth condition. What excites us is that in [8] Zhou with his coauthors investigated almost sure exponential stability of the backward Euler–Maruyama discretization (BEM) for highly nonlinear stochastic functional differential equations under a simple class of polynomial growth condition. The present work is a continuation of the literature [8], which extends the polynomial growth condition on the drift and diffusive coefficients in the model so as to adapts more nonlinear SFDEs.

Under the extended polynomial growth condition for our nonlinear model, in which both the drift and diffusive coefficients may defy the widely used linear growth condition, this paper will address the following sequent questions: if the exact solution of the model is almost sure exponentially stable, may the numerical schemes preserve the same stability property? If so, which method works well for this purpose? Is it ok for the Euler–Maruyama method? Actually, Wu and Mao had concluded in [9] that the Euler–Maruyama method cannot preserve the stability in general for nonlinear SDDEs, let alone the nonlinear SFDEs, while [9] confirmed that the backward Euler–Maruyama scheme can reproduce the stability of the analytic solutions under the same conditions generally. We cannot help thinking that can the backward Euler–Maruyama scheme preserve almost sure exponential stability for nonlinear SFDEs under the extended polynomial growth condition? In the paper, we shall say yes to this question. As a special case, we will improve some criterion in literature by way.

It is worth pointing out that the used techniques in literature may not adapt to our model under the nonlinear assumption. Actually, in some literature, the almost sure exponential stability is actually derived from the moment stability by a combined application of the Chebyshev inequality and the Borel–Cantelli lemma. For example, the authors of [11,12] proved that the moment exponential stability implies almost sure exponential stability under the linear growth by this technique, which is obviously unavailable for us under the nonlinear assumption; Higham with his coauthors developed another technique, that is the law of large numbers, in [11,13] to study the numerical sequence and obtained their almost sure stability criterion. Likewise, it is difficult for us to verify the condition for the application of this law under our nonlinear assumption due to the nonlinearity and hysteresis for the dynamics in the model. Thus, we had to turn to another way, namely that by the nonnegative semi-martingale convergence theorem, see [10]. Of course, this is also a typical way for us to achieve almost sure exponential stability. In principle, we need only to verify the $\mathcal{M}^2$-integrability condition for following this way, c.f.[1] etc, while this kind of condition is very easy to satisfy under even the highly nonlinear condition. To carry out the required results conveniently, the classical nonnegative semi-martingale convergence theorem is reformed to a practical version, namely the so called practical nonnegative semi-martingale boundedness lemma in the paper as preliminary. Besides, we will also discuss the global existence of the solutions of the underlying equation as a base for our stability issue as [8] did. We complete this by a new technique, the Fatou's Lamma for the first time in literature.

The rest of the paper is arranged as follows. In Section 2, we introduce some notations, assumptions and the preliminary lemmas. In Section 3, we study the stability of the analytic solution of the underlying SFDEs. In Section 4, we investigate the almost sure exponential stability of the backward Euler scheme. In Section 5, we provide a numerical example to illustrate the application of the obtained results.

2. Notations, model, assumptions and lemmas

Throughout this paper, we use the following notations. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, that is, it is right continuous and increasing while $\mathcal{F}_0$ contains all P-null sets.