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Trigonometric sums by Hermite interpolations

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ABSTRACT

This paper is devoted to the derivation of exact formulas for trigonometric sums at different nodes. These sums are obtained by the use of Hermite interpolations whose nodes are basically zeros of Chebyshev polynomials of the first and second kinds. We prove that some trigonometric sums are integer-valued and we derive some asymptotic formulas.

1. Introduction

In his paper, [20], Riesz proved that if f_n is a trigonometric polynomial of degree at most n, then for $\theta \in \mathbb{R}$,

$$f_n'(\theta) = \frac{1}{4n}\sum_{k=1}^{2n}(-1)^{k-1}\frac{f_n(\theta+\theta_k)}{\sin^2(\theta_k/2)}, \qquad \theta_k = \frac{(2k-1)\pi}{2n}, \tag{1.1}$$

see also [24, p. 10]. Letting $f_n(\theta)=\sin\theta$, $f_n(\theta)=\sin n\theta$, and letting $\theta=0$, (1.1) becomes the integer-valued trigonometric sums (IVTSs)

$$\sum_{k=1}^{n}(-1)^{k-1}\cot\left(\frac{(2k-1)\pi}{4n}\right) = n, \quad \sum_{k=1}^{n}\cot^2\left(\frac{(2k-1)\pi}{4n}\right) = 2n^2-n, \tag{1.2}$$

respectively. Noting this connection between trigonometric interpolation and IVTSs, Byrne and Smith derived in [3] more generalized IVTSs

$$\sum_{k=1}^{n}(-1)^{k-1}\cot^{2m-1}\left(\frac{(2k-1)\pi}{4n}\right) = \sum_{j=1}^{m}a_{m,j}n^{2j-1}, \tag{1.3}$$

where

$$a_{m,m} = \frac{2^{2m-2}}{(2m-2)!}E_{2m-2}, \quad a_{m,m-1} = -\frac{(2m-1)2^{2m-4}}{3(2m-4)!}E_{2m-4}, \quad \ldots,$$

see [3] for more details. Here, E_{2j} are the even-indexed Euler numbers defined by

$$\sec x = \sum_{j=0}^{\infty}E_{2j}\frac{x^{2j}}{(2j)!}, \qquad |x| < \frac{\pi}{2}.$$

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The right-hand side of (1.3) is an integer-valued polynomial where by an integer-valued polynomial $p(x)$ we mean that $p(x)$ is an integer whenever x is an integer.

In [2,15], the authors have implemented another approach to derive new IVTSs. They have used the sampling theorems associated with Sturm–Liouville difference operators, i.e. Lagrange-type interpolation formulas, cf. [1]. They evaluated a large number of trigonometric sums. Some simple sums in [15] are

$$\sum_{k=0}^{n-1}\sec^2\frac{(2k+1)\pi}{2(2n+1)} = \frac{2n^2+2n}{3},$$
$$\sum_{k=1}^{n-1}(-1)^{k+1}\csc^2\frac{k\pi}{2n} = \frac{1+2n^2}{6}+\frac{(-1)^n}{2},$$
$$\sum_{k=1}^{n-1}(-1)^{k+1}\tan^2\frac{k\pi}{2n} = (-1)^n\frac{1+2n^2}{6}+\frac{1}{2}-\begin{cases}1, & \text{if } n \text{ is even},\\ 0, & \text{if } n \text{ is odd}.\end{cases}$$

There are other ways to derive and apply trigonometric sums in mathematical and physical problems. A very efficient technique is the use of the theory of residues, as is established in [4,8–10]. The use of partial fractions also led to variety of trigonometric sums, as derived and exhibited in [6], for the nodes $\frac{k\pi}{n}, n\equiv 1(\mathrm{mod}2)$, $\frac{k\pi}{n}, n\equiv 0(\mathrm{mod}2)$ and $\frac{(2k+1)\pi}{n}, n\equiv 0(\mathrm{mod}4)$. See also [7].

This paper introduces a new method to derive new IVTSs which is expected to enrich the derivations of trigonometric sums. Although we pursue applying the interpolation theory to derive trigonometric sums, we implement neither Lagrange interpolations nor the trigonometric interpolations. Instead, we apply the Hermite interpolations to establish new trigonometric sums at different nodes.

The paper is organized as follows. In Section 2 we state the basic Hermite interpolation formulas that will be implemented throughout the paper. Section 3 contains direct applications to give explicit integer-valued sums. In Section 4 we obtain trigonometric sums that are not explicit, however the powers of the trigonometric functions are more general and asymptotic formulas are rigorously proved. In applications, as indicated in [4], exact closed forms of the trigonometric sums are not required, but only asymptotic formulas are needed.

Before leaving this introduction, we would like to mention some applications and motivations that emphasize the importance of deriving trigonometric sums. The exact formulas obtained for trigonometric sums lead to explicit evaluation of eigenvalues and integer eigenvalues of matrices with trigonometric entries, see [5,6]; solutions of probability problems, see [16,17,23], exact values of power series involving Euler and Bernoulli numbers, [13], as well as several physical problems as in [11,14,18]. See also [12] for applications of closed form of trigonometric sums associated with discrete Sturm–Liouville formulation of the discrete Gelfand–Yaglom theorem. Therefore, the derivations of new trigonometric sums at different nodes will increase our ability to deal with such applications.

2. Hermite interpolation formulae

This section involves some preliminary results. We introduce a Hermite interpolation formula and the major four cases that will lead to the trigonometric sums. Given distinct nodes $x_1,\dots,x_n$ in $[-1,1]$, real numbers $y_1',\dots,y_n'$, and a function $f(x)$ defined on $[-1,1]$, then the polynomial p_{2n-1} of degree at most $2n-1$ satisfying that

$$\begin{aligned} p_{2n-1}(x_j) &= f(x_j) = y_j,\\ p_{2n-1}'(x_j) &= y_j', \qquad j=1,\dots,n, \end{aligned} \tag{2.1}$$

is given in [21, pp. 23–24] by

$$p_{2n-1}(x) = \sum_{j=1}^{n} y_j\, h_j(x) + \sum_{j=1}^{n} y_j'\, q_j(x), \tag{2.2}$$

where,

$$\begin{aligned} h_j(x) &= \left(1-\frac{\omega''(x_j)}{\omega'(x_j)}(x-x_j)\right)l_j^2(x),\\ q_j(x) &= \big(x-x_j\big)l_j^2(x),\\ \omega(x) &= \prod_{j=1}^{n}\big(x-x_j\big),\\ l_j(x) &= \prod_{k=1,k\neq j}\left(\frac{x-x_k}{x_j-x_k}\right), \qquad j=1,\dots,n. \end{aligned} \tag{2.3}$$