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New cubic B-spline approximation for solving third order Emden–Flower type equations

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A B S T R A C T

In this article, the typical cubic B-spline collocation method equipped with new approximations for second and third order derivatives is employed to explore the numerical solution of a class of third order non-linear singular boundary value problems. The singularity is removed by means of L'Hospital's Rule. The Taylor's series expansion of the error term reveals that our new scheme is fifth order accurate. The proposed technique is tested on several third order Emden–Flower type equations and the numerical results are compared with those found in the current literature. It is found that our new approximation technique performs superior to the existing methods due to its simple implementation, straight forward interpolation and very less computational cost.

1. Introduction

Singular boundary value problems (BVP's) have earned a significant amount of research work in recent years due to their vide range of applications in different areas of science and engineering such as the study of boundary layer theory, theory of elastic stability, electro-hydrodynamics, thermal explosions, chemical reactions, beam theory, gravity-assisted flows, inelastic flows, electrically charged fluid flows, atomic nuclear reactions and astrophysics [1]. Consider the following class of 3*rd* order non-linear singular BVP's having singularity at $x = a$

$$\epsilon u'''(x) + \frac{m}{x}u''(x) + g(x)u'(x) = f(x,u), \quad a \le x \le b \tag{1.1}$$

subject to one of the following initial/boundary conditions

$$\begin{cases} u(a) = \alpha, & u'(a) = \beta, & u(b) = \gamma \\ u(a) = \alpha, & u'(a) = \beta, & u'(b) = \gamma \\ u(a) = \alpha, & u'(a) = \beta, & u''(a) = \gamma \end{cases} \tag{1.2}$$

where $0 < \epsilon \le 1$, $m \ge 1$ and α, β, γ are constants. Particularly, for $\epsilon = 1$, problem (1.1) is known as Emden–Flower type equation. The presence of singularity and small perturbation constant affect the convergence of direct numerical and analytical techniques. Therefore this type of problems has always remained challenging for the researchers. Khuri [2] developed a decomposition method for numerical solution of generalized Emden–Flower type equations. The series solution to higher order singular boundary value problems (SBVP's) has been investigated in [3–5] using modified Adomian decomposition method.

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Li et al. [6] used the Reproducing Kernel method for solving a class of 3rd order BVP's with non-linear end conditions. Aruna and Kanth [7] explored the series solution to third and fourth order non-linear SBVP's using Differential transformation method. Mahalik and Mohapatra [8] used adaptive grid generation algorithm together with Newton's Divided difference operator for solving singularly perturbed BVP's. Taiwo and Hassan [9] developed two numerical methods i.e. Iterative decomposition method and Bernstein polynomial method for solving third and fourth order non-linear SBVP's. Wazwaz [10] investigated the numerical solution of two kinds of third order Emden–Flower type equations using Variational iteration method. Wazwaz and Khuri [11] employed the Variational iteration method for solving Volterra integro-differential forms of Emden–Flower and Lane Emden type equations. El-Zahar [12] employed the Differential transformation method for approximating the piecewise analytic solution to a class of third order singularly perturbed BVP's with discontinuous source term.

The spline functions have been frequently employed for solving BVP's. These functions not only maintain high degree of smoothness at the knots but also possess the flexibility to approximate the solution at any point of the domain with high accuracy. The use of spline functions for numerical solution of BVP's was first introduced by Brickley [13], followed by Albasiny and Hoskins [14]. The fourth order accuracy of cubic spline interpolation scheme was proved by Fyfe [15]. The B-spline functions have been used to explore the numerical solution of a class of second order SBVP's in [16–21]. Khuri and Sayfy [22] proposed the adaptive cubic spline collocation method for solving second order Emden–Flower type equations.

In recent years, Caglar et al. [23] utilized fourth order B-spline functions for numerical solution to third order non-linear ordinary differential equations. Al-Said and Noor [24] employed third degree uniform polynomial splines for solving a system of third order BVP's. Khan and Aziz [25] investigated the numerical solution of third order linear and non-linear BVP's by means of Quintic splines. Rashidinia and Ghasemi [26] used the sextic splines for the numerical solution of higher order non-linear BVP's. Akram [27] proposed the Quartic polynomial spline for approximate solution to a class third order self adjoint singularly perturbed ordinary differential equations. The fourth degree B-spline collocation method was employed by Mishra et al. [28] for solving a class of third order singularly perturbed BVP's.

In this work, the numerical solution to a class of third order non-linear SBVP's is explored by means of third degree B-spline functions. The presented scheme is based on new approximations for the second and third order derivatives. The approximation for second order derivative is formulated first using proper linear combinations of the typical B-spline approximation for $u''(x)$ [29] at the neighbouring values and then this new approximation is further used to develop the approximation for third order derivative. In the last two decades, several numerical techniques have been used to explore the numerical solution to non-linear SBVP's but so far as we know, this new approximation has never been used for this purpose before. This work is presented as follows.

In Section 2, we shall present some basic results of typical third degree B-spline interpolation. The formulation of new approximation to $u''(x)$ and $u'''(x)$ is described in Sections 3 and 4, respectively. The numerical method is demonstrated in Section 5. The stability of the proposed numerical scheme is discussed in Section 6. In Section 7, an error analysis is presented in order to show that the proposed numerical scheme is fifth order accurate. The numerical results are given in Section 8.

2. Cubic B-spline functions

Let n be a positive integer and $a = x_0 < x_1 < x_2 < \dots < x_n = b$ be a uniform partition of $[a, b]$ having equidistant knots $x_i = x_0 + ih, i \in \mathbb{Z}$ using $h = \frac{b-a}{n}$. The typical third degree B-spline basis functions are defined as [29]

$$B_i(x) = \frac{1}{6h^3}\begin{cases} (x-x_{i-2})^3, & x \in [x_{i-2}, x_{i-1}] \\ h^3 + 3h^2(x-x_{i-1}) + 3h(x-x_{i-1})^2 - 3(x-x_{i-1})^3, & x \in [x_{i-1}, x_i] \\ h^3 + 3h^2(x_{i+1}-x) + 3h(x_{i+1}-x)^2 - 3(x_{i+1}-x)^3, & x \in [x_i, x_{i+1}] \\ (x_{i+2}-x)^3, & x \in [x_{i+1}, x_{i+2}] \\ 0, & \text{otherwise} \end{cases} \tag{2.1}$$

where $i = -1, 0, 1, 2, \dots, n+1$. Let $u(x)$ be a sufficiently smooth function, then there exists a unique cubic spline $U(x)$, satisfying the following interpolating conditions

$$\begin{aligned} U(x_i) &= u(x_i), \ \forall\ i = 0, 1, 2, \dots, n \\ U'(a) &= u'(a), \quad U'(b) = u'(b) \\ U''(a) &= u''(a), \quad U''(b) = u''(b) \end{aligned}$$

such that

$$U(x) = \sum_{i=-1}^{n+1} c_i B_i(x) \tag{2.2}$$

where c_i's are finite constants which are to be determined. For the sake of simplicity, we denote the B-spline approximations $U(x_j)$, $U'(x_j)$ and $U''(x_j)$ by U_j, m_j and M_j respectively. The cubic B-spline basis functions (2.1) and (2.2) give the following relations

$$U_j = \sum_{i=j-1}^{j+1} c_i B_i(x) = \frac{1}{6}\left(c_{j-1} + 4c_j + c_{j+1}\right) \tag{2.3}$$