Applied Mathematics and Computation 331 (2018) 358–377

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Joint moments of the total discounted gains and losses in the renewal risk model with two-sided jumps

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ARTICLE INFO

Keywords:
Renewal risk model
Two-sided jumps
Joint moments
Total discounted claims/gains
Number of downward/upward jumps

ABSTRACT

This paper considers a renewal insurance risk model with two-sided jumps (e.g. Labbé et al., 2011), where downward and upward jumps typically represent claim amounts and random gains respectively. A generalization of the Gerber–Shiu expected discounted penalty function (Gerber and Shiu, 1998) is proposed and analyzed for sample paths leading to ruin. In particular, we shall incorporate the joint moments of the total discounted costs associated with claims and gains until ruin into the Gerber–Shiu function. Because ruin may not occur, the joint moments of the total discounted claim costs and gain costs are also studied upon ultimate survival of the process. General recursive integral equations satisfied by these functions are derived, and our analysis relies on the concept of 'moment-based discounted densities' introduced by Cheung (2013). Some explicit solutions are obtained in two examples under different cost functions when the distribution of each claim is exponential or a combination of exponentials (while keeping the distributions of the gains and the inter-arrival times between successive jumps arbitrary). The first example looks at the joint moments of the total discounted amounts of claims and gains whereas the second focuses on the joint moments of the numbers of downward and upward jumps until ruin. Numerical examples including the calculations of covariances between the afore-mentioned quantities are given at the end along with some interpretations.

1. Introduction

1.1. The model, quantities of interest and literature review

In this paper, the renewal risk process with two-sided jumps is used to model the evolution of an insurance company's surplus over time (e.g. [24]). Specifically, the surplus level of the insurance company at time t is given by

$$U(t) = u + ct + \sum_{i=1}^{N(t)} Y_i, \qquad t \geq 0, \tag{1.1}$$

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https://doi.org/10.1016/j.amc.2018.03.037

where $u = U(0) \geq 0$ is the initial surplus, $c \geq 0$ is the (net) premium income per unit time that is assumed to be a constant, $\{N(t)\}_{t\geq 0}$ is a renewal process that counts the number of jumps, and $\{Y_i\}_{i=1}^{\infty}$ forms an independent and identically distributed (i.i.d.) sequence representing the jump sizes that may be positive or negative. The Y_i's are assumed to be continuous random variables with common density $p(\cdot)$. In addition, the counting process $\{N(t)\}_{t\geq 0}$ is characterized by the sequence of arrival epochs $\{T_i\}_{i=1}^{\infty}$ such that $N(t) = \sup\{i \in \mathbb{N} : T_i \leq t\}$ (where $\mathbb{N}$ is the set of non-negative integers). The corresponding inter-arrival times $\{V_i = T_i - T_{i-1}\}_{i=1}^{\infty}$ (with the definition $T_0 = 0$) form an i.i.d. sequence of positive continuous random variables with common density $k(\cdot)$. Furthermore, the sequences $\{Y_i\}_{i=1}^{\infty}$ and $\{V_i\}_{i=1}^{\infty}$ are assumed to be mutually independent. The time of ruin of the surplus process (1.1) is defined as $\tau = \inf\{t \geq 0 : U(t) < 0\}$ with the convention that $\tau = \infty$ if $U(t) \geq 0$ for all $t \geq 0$.

Because each jump size Y_i in the insurance risk process (1.1) may take on positive or negative values, in our analysis it will be more convenient to separate the contributions of upward and downward jumps. To this end, we define $N_+(t) = \sum_{i=1}^{N(t)} 1\{Y_i > 0\}$ to be the number of upward jumps until time t (where $1\{A\}$ is the indicator function of an event A), which is related to the arrival epochs $\{T_{+,i}\}_{i=1}^{\infty}$ of the upward jumps via $N_+(t) = \sup\{i \in \mathbb{N} : T_{+,i} \leq t\}$. Similarly, let $N_-(t) = \sum_{i=1}^{N(t)} 1\{Y_i < 0\} = \sup\{i \in \mathbb{N} : T_{-,i} \leq t\}$ with $\{T_{-,i}\}_{i=1}^{\infty}$ being the arrival epochs of the downward jumps. For each $i \in \mathbb{N}^+$ (where $\mathbb{N}^+$ is the set of positive integers), there is a unique $j \in \mathbb{N}^+$ such that $T_{+,i} = T_j$ for which we let $Y_{+,i} = Y_j$ be the size of the ith upward jump. Analogously, we define $Y_{-,i} = -Y_j$ to be the magnitude of the ith downward jump if $T_{-,i} = T_j$. Hence, the surplus process (1.1) can be rewritten as

$$U(t) = u + ct - \sum_{i=1}^{N_-(t)} Y_{-,i} + \sum_{i=1}^{N_+(t)} Y_{+,i}, \qquad t \geq 0. \tag{1.2}$$

It is assumed that $\Pr\{Y_1 > 0\} = q_+ \geq 0$ and $\Pr\{Y_1 < 0\} = q_- = 1 - q_+ > 0$. (We exclude the case $q_- = 0$ which results in monotonically increasing sample paths.) Then for $y > 0$ the common densities of the sequences $\{Y_{+,i}\}_{i=1}^{\infty}$ and $\{Y_{-,i}\}_{i=1}^{\infty}$ are given by $p_+(y) = (d/dy)\Pr\{Y_1 \leq y | Y_1 > 0\}$ and $p_-(y) = (d/dy)\Pr\{-Y_1 \leq y | Y_1 < 0\}$ respectively, and one can write

$$p(y) = q_+ p_+(y) 1\{y > 0\} + q_- p_-(-y) 1\{y < 0\}.$$

Furthermore, the positive security loading condition $cE[V_1] + E[Y_1] = cE[V_1] + q_+E[Y_{+,1}] - q_-E[Y_{-,1}] > 0$ is assumed to hold, and this ensures that the ruin probability $\psi(u) = \Pr\{\tau < \infty | U(0) = u\}$ is strictly less than 1 for all $u \geq 0$ (e.g. [2], Chapter III, Corollary 3.2(a)). Note that the random sums $\sum_{i=1}^{N_-(t)} Y_{-,i}$ and $\sum_{i=1}^{N_+(t)} Y_{+,i}$ in (1.2) are independent if $\{N(t)\}_{t\geq 0}$ is a Poisson process, but this is generally not true when $\{N(t)\}_{t\geq 0}$ is a renewal process. In addition, $\{N_+(t)\}_{t\geq 0}$ (resp. $\{N_-(t)\}_{t\geq 0}$) is a renewal process with i.i.d. inter-arrival times having common density $\sum_{i=1}^{\infty} q_-^{i-1} q_+ k^{*i}(\cdot)$ (resp. $\sum_{i=1}^{\infty} q_+^{i-1} q_- k^{*i}(\cdot)$), where $k^{*i}(\cdot)$ is the i-fold convolution density of $k(\cdot)$.

In the risk process (1.1) or (1.2), an upward jump is regarded as a stochastic income and a downward jump is the result of a loss. While it is clear that losses or insurance claims usually arise from property and casualty insurance business, random income arises for life annuity or pension funds in which the insurer pays annuities to its policyholders and earns part of the reserves when a policyholder dies (e.g. [30], p.116). Therefore, the model in the present paper can be suitable for insurance companies with business in both property and casualty insurance and life annuities. Apart from the above interpretation, corrections of previous overstatement of losses can also be another source of random income. For the rest of the paper, we shall use the terminologies 'loss', 'claim' and 'downward jump' interchangeably, and the same applies to the words 'gain' and 'upward jump'. There have been a number of studies concerning risk models with two-sided jumps in recent years. Much of the pertinent literature has focused on the analysis of the Gerber–Shiu expected discounted penalty function [21], which is defined as

$$\phi_{\delta_1}(u) = E[e^{-\delta_1 \tau} w(U(\tau^-), |U(\tau)|) 1\{\tau < \infty\} | U(0) = u], \qquad u \geq 0. \tag{1.3}$$

Here $\delta_1 \geq 0$ can be regarded as a force of interest or the Laplace transform argument of τ, and $w(\cdot,\cdot)$ is the so-called 'penalty' as a non-negative function of the surplus immediately before ruin $U(\tau^-)$ and the deficit at ruin $|U(\tau)|$. Under the compound Poisson model without deterministic premium income (i.e. $c = 0$), [25] and [1] derived the solution of the Gerber–Shiu function (and its special case where $w(\cdot,\cdot)$ only depends on $|U(\tau)|$) under specific distributional assumptions on either downward or upward jumps. [6], (Section 8) also contains some explicit results under two-sided exponential jumps as well as an application of a Gerber–Shiu function to price perpetual American put options. For the risk model (1.1) with arbitrary inter-arrival times, [8] and [24], respectively considered the cases $c < 0$ and $c \geq 0$, and some structural properties of the Gerber–Shiu function in connection with defective renewal equations are given. Interested readers are referred to [3,5,7,22,29] for the study of first passage times and one-sided or two-sided exit problems in related models with two-sided jumps.

While the Gerber–Shiu function (1.3) serves as a powerful tool that unifies the study of τ, $U(\tau^-)$ and $|U(\tau)|$, it is instructive to note that the random variables $U(\tau^-)$ and $|U(\tau)|$ are defined at the ruin time τ. This led various researchers to analyze generalizations of the Gerber–Shiu function (mainly in risk models without upward jumps). In particular, additional random variables defined before the time of ruin are incorporated into the penalty function (or the joint law underlying the Gerber–Shiu function), such as the minimum surplus level before ruin (e.g. [4,16] and [11]), the maximum surplus before ruin (e.g. [23], and [10]), the surplus level immediately after the second last claim before ruin (see e.g. [11,32,33], and [34]). On the other hand, [26] and [20] included the number of claims until ruin in the Gerber–Shiu function in the form of a