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On the dynamics of Kopel's Cournot duopoly model

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ABSTRACT

In this paper we study the duopoly model proposed by M. Kopel [26], where two firms compete "*a la Cournot*" and the reaction curves have a higher degree of coupling in the sense that firms have to make their choices simultaneously. We will make a descriptive analysis of the two-dimensional model, making an approach through particular situations. On the other hand, when the firms are homogeneous a one-dimensional invariant subset is present in the model. We will give an analytical proof of the existence of a (topological) chaotic behavior for a wide range of parameter values and we will study when chaotic synchronization and collusion occur.

1. Introduction

M. Kopel introduced in [18] a duopoly model in which two firms compete "*a la Cournot*". This model has been widely studied from different points of view in several papers [2,16,18,25]. Thus, in [16] the stability of the fixed points is analyzed and in [25] the topological horseshoe theory is used to detect topological chaos.

We will use both theoretical and numerical techniques to give a detailed overview of the model. The dynamics is approached using different techniques. Although the model is two-dimensional, the knowledge and study of some particular cases, for example when $\lambda = 1$, can shed light on the global dynamics of the model. In particular, we study the invariant set $\Delta = \{(x, y) \in \mathbb{R}^2 : x = y\}$, which can be deeply analyzed because it is reduced to a one-dimensional map, and we may use different well-known facts to describe the model. We also analyze when firms synchronize to Δ, and when such synchronization is chaotic. For the general two-dimensional model the attractors, their number and layout, are considered in a descriptive way and some related properties are given. A remarkable feature is that this model can be derived from different economic assumptions, see [18]. This is a particularly important fact from the point of view of benefits since the results change depending on the economic settings, the prices and demand conditions. We consider both alternative options when we study the average profit and the market situation along the time. The average profit along the chaotic orbits is compared to the profits obtained in the stationary points for both firms. In particular, we study the cases in which both firms are interested in making agreements in order to improve their profits and collusion is the best option for both firms.

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2. Analytic description of the model

In this section we present the equations of the model. The model of Kopel uses cost functions with externalities depending not only by the own firm's production but also on the production of the other firm. These externalities turn out in a model with non-monotonic reaction curves for both firms, in this particular case, the quadratic map $l_\mu(x) = \mu x(1-x)$. Two firms X and Y, plan their productions, namely x_t, y_t, at time t following the system of difference equations:

$$\begin{aligned} x_{t+1} &= (1-\lambda_1)x_t + \lambda_1\,\mu_1 y_t\,(1-y_t) \\ y_{t+1} &= (1-\lambda_2)y_t + \lambda_2\,\mu_2 x_t\,(1-x_t), \end{aligned} \tag{1}$$

where $\lambda_i \in [0, 1]$ and $\mu_i > 0$ for $i = 1, 2$. The positive parameters μ_i measure the intensity of the effect that one firm's actions has on the other firm (the extent of the externality). Firms choose a weighted average between the previous production and the computed one with weights $1-\lambda_i$ and λ_i, respectively. When $\lambda = 1$, the strategy is played in a sequential process, in which each firm decides its production observing the production of the another firm, whereas $0 < \lambda < 1$ firms make their choices according an adaptive process in which x_{t+1} (y_{t+1}) is a weighted combination between its previous output, x_t (y_t) and $l_{\mu_1}(y_t)$ $(l_{\mu_2}(x_t))$. In this process the parameters λ_1 and λ_2 can be considered as measures of the speed of the adjustment of each firm. Since productions cannot be negative, we prefer to rewrite the model as

$$\begin{aligned} x_{t+1} &= \max\{0, (1-\lambda_1)x_t + \lambda_1\,\mu_1 y_t\,(1-y_t)\} \\ y_{t+1} &= \max\{0, (1-\lambda_2)y_t + \lambda_2\,\mu_2 x_t\,(1-x_t)\}, \end{aligned} \tag{2}$$

The two-dimensional map

$$R_{\lambda_1,\lambda_2,\mu_1,\mu_2}(x, y) := (\max\{0, (1-\lambda_1)x + \lambda_1\,\mu_1 y\,(1-y)\}, \max\{0, (1-\lambda_2)y + \lambda_2\,\mu_2 x\,(1-x)\}),$$

describes the simultaneous choices of both firms and the sequence (x_t, y_t) is given by

$$(x_t, y_t) := R^t_{\lambda_1,\lambda_2,\mu_1,\mu_2}(x_0, y_0) = \overbrace{R_{\lambda_1,\lambda_2,\mu_1,\mu_2} \circ \cdots \circ R_{\lambda_1,\lambda_2,\mu_1,\mu_2}}^{t \text{ times}}(x_0, y_0),$$

where (x_t, y_t) denotes the t-iterated point of the orbit with initial condition (x_0, y_0).

In general, authors (see [2,16,18,25]) consider a restricted model in which $\lambda_1 = \lambda_2 = \lambda$, $\mu_1 = \mu_2 = \mu$. The two restrictions according to which $\lambda_1 = \lambda_2 = \lambda$ and $\mu_1 = \mu_2 = \mu$ define the condition for a symmetric duopoly model in which the strategic relationship between the output offers of the two firms is non-monotonic. For each firm the range of potential output offers put forward by its opponent is normalized in the interval [0, 1] and the optimal output reply belong to the interval $[0, \frac{\mu}{4}]$. Thus, μ can be interpreted as an indicator of production capacity. The reaction map takes the form

$$R_{\lambda,\mu}(x, y) := (\max\{0, (1-\lambda)x + \lambda\,\mu y\,(1-y)\}, \max\{0, (1-\lambda)y + \lambda\,\mu x\,(1-x)\}), \tag{3}$$

and if

$$f_{\lambda,\mu}(x, y) = (1-\lambda)x + \lambda\,\mu\,y(1-y) \tag{4}$$

the reaction map (3) can be rewritten as

$$R_{\lambda,\mu}(x, y) := (\max\{0, f_{\lambda,\mu}(x, y)\}, \max\{0, f_{\lambda,\mu}(y, x)\}). \tag{5}$$

The parametric space is defined as

$$\Omega = \{(\lambda, \mu) \in \mathbb{R}^2 : \lambda \in [0, 1] \text{ and } \mu > 0\}. \tag{6}$$

In the general case, algebraic programs do not work properly to give results, but also the restricted model presents a very rich dynamics by itself. Observe that now the set $\Delta = \{(x, x) : x \geq 0\}$ is invariant, that is $R_{\lambda,\mu}(x, x) \in \Delta$ for each $(\lambda, \mu) \in \Omega$. The main consequence of this fact is that we can study the dynamics restricted to Δ reducing the equations to a one-dimensional model, where dynamics can be studied using a wider variety of techniques. In fact, we will study the restricted model as well, but we will make some approaches to the general one by computing the topological entropy with prescribed accuracy in some particular cases. It means that we can prove the existence of topological chaos not only simple numerical simulations but making an estimation up to prescribed error. It is worth to mention that the existence of topological chaos is not always physically observable using simulations.

3. The invariant diagonal set: one-dimensional dynamics

This section is devoted to study the dynamics when both firms (which are equal in terms of plan their choices) produce the same quantities at the beginning. The invariant set $\Delta = \{(x, x) : x \geq 0\}$ can be seen as an interval of the real line and the dynamics can be studied in terms of a one-dimensional map. Observe that Δ is included in the two-dimensional phase space and we are interested in checking whether the dynamics outside Δ converges to the dynamics on Δ, with a special emphasis in the existence of chaotic synchronization. The one-dimensional dynamics is given by the function

$$f_{\lambda,\mu}(x, x) = \max\{0, (1-\lambda)x + \lambda\,\mu x(1-x)\},$$