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An ε-uniform numerical method for third order singularly perturbed delay differential equations with discontinuous convection coefficient and source term

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ABSTRACT

A class of third order singularly perturbed Boundary Value Problems (BVPs) for ordinary delay differential equations with discontinuous convection–diffusion coefficient and source term is considered in this paper. The existence and uniqueness of the solution has been proved. Further, a fitted finite difference method on Shishkin mesh is suggested to solve the problem. Numerical solution converges uniformly to the exact solution. The order of convergence of the numerical method presented here is of almost first order. Numerical results are provided to illustrate the theoretical results.

1. Introduction

In this paper, we study the following third order singularly perturbed delay differential equations with discontinuous convection coefficient and source term:

Find $u \in Y = C^1(\overline{\Omega}) \cap C^2(\Omega) \cap C^3(\Omega^*)$ such that

$$\begin{cases} -\varepsilon u'''(x) + a(x)u''(x) + b(x)u'(x) + c(x)u(x) + d(x)u'(x-1) = f(x), & x \in \Omega^*, \\ u(x) = \phi(x), \ x \in [-1, 0], \ u'(2) = l, \ \phi \in C^1[-1, 0], \end{cases} \tag{1.1}$$

where $0 < \varepsilon \ll 1$, $a(x) = \begin{cases} a_1(x), & x \in [0, 1], \\ a_2(x), & x \in (1, 2], \end{cases}$ $f(x) = \begin{cases} f_1(x), & x \in [0, 1], \\ f_2(x), & x \in (1, 2], \end{cases}$ $a_1(1-) \neq a_2(1+)$, $f_1(1-) \neq f_2(1+)$, $a_i(x) \geq \alpha_i > \alpha + 2 > 3$, $i = 1, 2$, $b(x) \geq \beta_0 \geq 0$, $\gamma_0 \leq c(x) \leq \gamma \leq 0$, $\eta_0 \leq d(x) \leq 0$, $2\alpha + 24\gamma_0 + 5\eta_0 > 0$ and b, c, d are sufficiently smooth on $\overline{\Omega}$, a and f are sufficiently smooth and bounded on Ω^*, $\Omega^* = \Omega^- \cup \Omega^+$, $\Omega^- = (0, 1)$, $\Omega^+ = (1, 2)$, $\Omega = (0, 2)$.

Existence, Oscillation and boundedness of the solution have been well studied in [1–5] and the reference therein for third order delay differential equations with smooth data. As far as our knowledge goes, existence of the solution for third order delay differential equations with non smooth data is not available in the literature. In this paper, we proved existence and uniqueness of the solution for problem (1.1).

It is well known that an efficient way of handling the above problem (1.1) from a numerical method perspective is employing a layer-adapted mesh, such as a Shishkin mesh or a Bakhvalov mesh. For the above problem, a layer adopted piecewise uniform Shishkin mesh is constructed and on this mesh a fitted finite difference method is suggested. Further, it

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is proved that, the numerical solution is converges uniformly to the solution. One may refer [6–8], for more details regarding to uniform convergence analysis for singular perturbation problems. Recently many authors have suggested some numerical methods for second order singularly perturbed delay differential equations. For instance, Geng and Qian [9] proposed and analyzed modified reproducing kernel method for the problem of convection diffusion type with non vanishing delay. Whereas, the authors in [10] suggested exponentially fitted initial value technique for convection diffusion problem with vanishing delay. In [11] and [12], the author suggested a fitted finite difference method for single and system of convection diffusion problem with discontinuous data, etc. Few authors have considered and suggested some uniformly valid numerical methods for singularly perturbed partial delay differential equations, to cite a few: [13–16]. For more details pertaining to the numerical methods for singularly perturbed delay differential equations, one may refer [9–16] and the references cited therein.

The above BVP (1.1) can be written as,

Find $\bar{u} = (u_1,\ u_2),\ u_1 \in Y_1 = C^0(\bar{\Omega}) \cap C^1(0,2]$ and $u_2 \in Y_2 = C^0(\overline{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^*)$ such that

$$P_1\bar{u}(x) := u_1'(x) - u_2(x) = 0,\ x \in \Omega \cup \{2\}, \tag{1.2}$$

$$P_2\bar{u}(x) := \begin{cases} -\varepsilon u_2''(x) + a_1(x)u_2'(x) + b(x)u_2(x) + c(x)u_1(x) = f_1(x) - d(x)\phi'(x-1),\ x \in \Omega^-, \\ -\varepsilon u_2''(x) + a_2(x)u_2'(x) + b(x)u_2(x) + c(x)u_1(x) + d(x)u_2(x-1) = f_2(x),\ x \in \Omega^+, \end{cases} \tag{1.3}$$

$$u_1(0) = \phi(0),\ u_2(0) = \phi'(0),\ u_2(1-) = u_2(1+),\ u_2'(1-) = u_2'(1+),\ u_2(2) = l,$$

where $u_2(1-)$ and $u_2(1+)$ denote the left and right limits of u_2 at $x = 1$, respectively.

The present paper is organized as follows. The existence of the solution of the problem stated in the present section is proved in the next section. A maximum principle for the DDE is established in Section 3. Further a stability result is derived. Derivative estimate for the solution of the problems is derived in Section 4. The present numerical method is described in Section 5 and an error estimate is derived in Section 6. Section 7 presents numerical examples. The paper is concluded with discussion (Section 8).

2. Existence of solution

In this section, we proved the existence of the solution of the problem stated above.

Theorem 2.1. *Problem (1.1) has a solution $\bar{u} = (u_1, u_2)$, where $u_1 \in C^0(\bar{\Omega}) \cap C^1(\Omega \cup \{2\})$ and $u_2 \in C^0(\bar{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^*)$.*

Proof. The proof is by construction. Let $\bar{y}_L = (y_{1L}, y_{2L})$ and $\bar{y}_R = (y_{1R}, y_{2R})$ be particular solutions of the following problems, respectively:

$$\begin{cases} y_{1L}'(x) - y_{2L}(x) = 0,\ x \in \Omega^-, \\ -\varepsilon y_{2L}''(x) + a_1(x)y_{2L}'(x) + b(x)y_{2L}(x) + c(x)y_{1L}(x) = f_1(x) - d(x)y_{2L}(x-1), x \in \Omega^-, \\ y_{1L}(x) = \phi(x),\ y_{2L}(x) = \phi'(x), x \in [-1,0] \end{cases} \tag{2.1}$$

and

$$\begin{cases} y_{1R}'(x) - y_{2R}(x) = 0,\ x \in \Omega^+, \\ -\varepsilon y_{2R}''(x) + a_2(x)y_{2R}'(x) + b(x)y_{2R}(x) + c(x)y_{1R}(x) = f_2(x) - d(x)y_{2L}(x-1), x \in \Omega^+. \end{cases} \tag{2.2}$$

Further, let $\bar{\phi}_1 = (\phi_{11}, \phi_{12})$, $\bar{\phi}_2 = (\phi_{21}, \phi_{22})$, $\bar{\phi}_3 = (\phi_{31}, \phi_{32})$ be the functions satisfy the following problems, respectively

$$\begin{cases} \phi_{11}'(x) - \phi_{12}(x) = 0,\ x \in \Omega \cup \{2\}, \\ -\varepsilon\phi_{12}''(x) + a_1(x)\phi_{12}'(x) + b(x)\phi_{12}(x) + c(x)\phi_{11}(x) + d(x)\phi_{12}(x-1) = 0,\ x \in \Omega, \\ \phi_{11}(x) = 0,\ x \in [-1,0], \phi_{12}(x) = 0,\ x \in [-1,0], \phi_{12}(2) = 1, \end{cases} \tag{2.3}$$

$$\begin{cases} \phi_{21}'(x) - \phi_{22}(x) = 0,\ x \in \Omega \cup \{2\}, \\ -\varepsilon\phi_{22}''(x) + a_2(x)\phi_{22}'(x) + b(x)\phi_{22}(x) + c(x)\phi_{21}(x) + d(x)\phi_{22}(x-1) = 0,\ x \in \Omega, \\ \phi_{21}(x) = 0,\ x \in [-1,0], \phi_{22}(x) = 0,\ x \in [-1,0], \phi_{22}(2) = 1, \end{cases} \tag{2.4}$$

$$\begin{cases} \phi_{31}'(x) - \phi_{32}(x) = 0,\ x \in \Omega \cup \{2\}, \\ -\varepsilon\phi_{32}''(x) + a_2(x)\phi_{32}'(x) + b(x)\phi_{32}(x) + c(x)\phi_{31}(x) + d(x)\phi_{32}(x-1) = 0,\ x \in \Omega, \\ \phi_{31}(x) = 0,\ x \in [-1,0], \phi_{32}(x) = 1,\ x \in [-1,0], \phi_{32}(2) = 0. \end{cases} \tag{2.5}$$

Now define $\bar{y} = (y_1, y_2)$,

$$y_1(x) = \begin{cases} y_{1L}(x) + A\phi_{11}(x), x \in \Omega^-, \\ y_{1R}(x) + \phi_{21}(x)[u_2(2) - y_{2R}(2)] + B\phi_{31}(x), x \in \Omega^+, \end{cases} \tag{2.6}$$