Applied Mathematics and Computation 331 (2018) 430–444

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A multiple-scale higher order polynomial collocation method for 2D and 3D elliptic partial differential equations with variable coefficients

Guojun Liu [a], Wentao Ma [a,*], Hailong Ma [b], Lin Zhu [a]

[a] School of Mathematics and Statistics, Ningxia University, Yinchuan 750021, China
[b] Institute of Ethnic Preparatory Education, Ningxia University, Yinchuan 750021, China

ARTICLE INFO

Keywords:
Elliptic PDEs
Collocation method
Multiple-scale higher order polynomial approximation
Mixed boundary conditions
Near-singular problem

ABSTRACT

In this paper, we present a multiple-scale higher order polynomial collocation method for the numerical solution of 2D and 3D elliptic partial differential equations (PDEs) with variable coefficients. The collocation method with higher order polynomial approximation is very simple for solving PDEs, but it has not become the mainstream method. The main reason is that its resultant algebraic equations have highly ill-conditioned behavior. In our scheme, the multiple-scale coefficients are introduced in the polynomial approximation to overcome the ill-conditioned problem. Based on the concept of the equilibrate matrix, the multiple scales are automatically determined by the collocation points. We find these scales can largely reduce the condition number of the coefficient matrix. Numerical results confirm the accuracy, effectiveness and stability of the present method for smoothed and near-singular 2D and 3D elliptic problems on various irregular domains.

1. Introduction

The elliptic partial differential equations (PDEs) have been widely used to model real physics phenomenon such as, the flow of air pollutions, temperature deflection, electrostatic potential, velocity potential, stream function, fluid flow [1,2], etc. We always encounter the elliptic PDEs with variable coefficients arising in the heterogeneous media. During the last decades, many numerical methods have been presented and developed in this field. Nowadays the meshless collocation methods attract many researchers and are the mainstream in these methods due to the advantages of simple algorithm, computational accuracy and truly meshless. The method fundamental solution (MFS) is a well-known boundary collocation meshless method. Fairweather and Karageorghs [3] directly used the MFS for solving elliptic boundary value problem. The MFS is an inherently meshless boundary method and has the property of exponential convergence. However, Golberg and Chen [4] and Chen et al. [5] point out that the MFS is an ill-conditioned problem when the number of source points is increasing and when the distances of source points are increased. Besides, the MFS has some difficulties for dealing with complicated geometrics with discontinuous boundary conditions. Many researchers have devoted their efforts to solve these problems in MFS, and the detail reports can be found in literature [7–10]. The method of particular solutions (MPS) is another popular meshless collocation method. In these approaches, a variety basis functions such as Bessel function [11],

* Corresponding author.
E-mail address: wt-ma2002@163.com (W. Ma).

https://doi.org/10.1016/j.amc.2018.03.021
0096-3003/

radial basis function (RBF) [12–15], polynomial function [16–19] and spline function [20] have been used. However, it is still a challenge to obtain a particular solution and the homogeneous solution is not always available.

The purpose of this paper is to develop a multiple-scale higher order polynomial meshless method, inheriting the advantage of easy numerical implementation and having higher accuracy and great flexibility in solving 2D and 3D elliptic PDEs with variable coefficients in arbitrary domain. We begin with the following 2D and 3D linear elliptic PDES of the general form:

$$\sum_{i=1}^{d}\left[a_i(\boldsymbol{x})\frac{\partial^2 u(\boldsymbol{x})}{\partial {x_i}^2}+b_i(\boldsymbol{x})\frac{\partial u(\boldsymbol{x})}{\partial x_i}\right]+c(\boldsymbol{x})u(\boldsymbol{x})=f(\boldsymbol{x}),\boldsymbol{x}\in\Omega\subset R^d,d=2,3 \tag{1}$$

with mixed boundary condition

$$u(\boldsymbol{x})=g(\boldsymbol{x}),\boldsymbol{x}\in\Gamma_1, \tag{2}$$

$$\frac{\partial u(\boldsymbol{x})}{\partial n}=h(\boldsymbol{x}),\boldsymbol{x}\in\Gamma_2 \tag{3}$$

where $\Gamma=\Gamma_1\cup\Gamma_2$, the coefficients $a_i(\boldsymbol{x})$,$b_i(\boldsymbol{x})$,$c(\boldsymbol{x})$ and $f(\boldsymbol{x})$,$g(\boldsymbol{x})$,$h(\boldsymbol{x})$ are enough smoothed functions, $a_i(\boldsymbol{x})$ provide the elliptic type of the PDE and $\Omega\subset R^d$ is the problem domain. Under the given Dirichlet boundary condition (2) and Neumann boundary condition (3), we must solve Eq. (1) to find the solution of $u(\boldsymbol{x})$.

It is well known that the polynomial basis function is easily to be implemented. Compared with RBF, the polynomial basis has no parameters to be optimized. However, the polynomial basis function is seldom used to solve the PDEs directly. The main reason is that the resultant algebraic equations are ill-conditioned and the numerical results are unstable when the order of polynomial approximation is becoming very high. Therefore, the higher order polynomial basis function is not ideal for a global approximation. In order to overcome those difficulties, Liu and Atluri [21] introduced a characteristic length into the higher-order polynomials expansion to solve the ill-conditioned linear interpolation problems. Liu and Kuo [22] and Liu and Young [23] proposed a multiple-scale Pascal polynomial triangular solving 2D elliptic equations, stokes and inverse Cauchy problems. Chang [24] proposed a multiple-scale Pascal triangle method to solve steady-state nonlinear heat conduction problems in arbitrary plane domain enclosed by a complex boundary shape. However, most previous works only consider PDEs with constant coefficients in 2D problem and the solutions are often relatively simple functions (such as polynomials, trigonometric functions, exponential functions, etc.) or their combination.

In this paper, we use very higher order polynomial approximation (order >10) for solving 2D and 3D elliptic PDEs with variable coefficients in arbitrarily complicated geometrics. Meanwhile, developing a multiple-scale technique based on the concept of equilibrate matrix to alleviate the ill-conditioned problem of the resultant algebraic equations. The organization of this paper is as follows: in Section 2, we derive the meshless collocation method with polynomial basis functions of higher order for 3D cases. Then we introduce a multiple-scale into the higher order polynomial collocation method according to the concept of equilibration matrix in Section 3. The numerical tests are to verify the performance of the proposed method in Section 4. The conclusions are addressed in Section 5.

2. Higher order polynomial collocation method

To discretize the given Eqs. (1)–(3), we straightforwardly employ the higher polynomial functions as the approximate solution. In 3D case, the approximate solution $u(\boldsymbol{x})$ can be expressed as

$$u(x_1,x_2,x_3)=\sum_{0\le i+j+k\le m}^{M} a_{ijk}{x_1}^i{x_2}^j{x_3}^k, \tag{4}$$

In which three superscripts i, j, k are positive integers, the coefficients a_{ijk} are to be decided. m is the highest order of the above polynomial and the total number of all elements is $M=(m+1)(m+2)(m+3)/6$. Obviously, Eq. (4) degenerates into the 2D form when $k=0$.

From Eq. (4), it is straightforward to write

$$\frac{\partial u(x_1,x_2,x_3)}{\partial x_1}=\sum_{0\le i+j+k\le m}^{M} a_{ijk}i{x_1}^{i-1}{x_2}^j{x_3}^k \tag{5}$$

$$\frac{\partial u(x_1,x_2,x_3)}{\partial x_2}=\sum_{0\le i+j+k\le m}^{M} a_{ijk}j{x_1}^{i}{x_2}^{j-1}{x_3}^k \tag{6}$$

$$\frac{\partial u(x_1,x_2,x_3)}{\partial x_3}=\sum_{0\le i+j+k\le m}^{M} a_{ijk}k{x_1}^{i}{x_2}^{j}{x_3}^{k-1} \tag{7}$$