Applied Mathematics and Computation 329 (2018) 23–37

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

High order well-balanced discontinuous Galerkin methods for Euler equations at isentropic equilibrium state under gravitational fields

Shouguo Qian[a], Yu Liu[a], Gang Li[a,*], Li Yuan[b]

[a] School of Mathematics and Statistics, Qingdao University, Qingdao, Shandong 266071, PR China
[b] LSEC & NCMIS, Institute of Computational Mathematics and Scientific/Engineering Computing, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, PR China

ARTICLE INFO

Keywords:
Euler equations
Isentropic equilibrium state
Discontinuous Galerkin methods
Well-balanced property
Gravitational fields

ABSTRACT

Euler equations under gravitational fields often appear in some interesting astrophysical and atmospheric applications. The Euler equations are coupled with gravitational source term due to the gravity and admit hydrostatic equilibrium state where the flux produced by the pressure gradient is exactly balanced by the gravitational source term. In this paper, we construct high order discontinuous Galerkin methods for the Euler equations under gravitational fields, which are well-balanced for the isentropic type hydrostatic equilibrium state. To maintain the well-balanced property, we first reformulate the governing equations in an equivalent form. Then we propose a novel source term approximation based on a splitting algorithm as well as well-balanced numerical fluxes. Rigorous theoretical analysis and extensive numerical examples all suggest that the proposed methods maintain the hydrostatic equilibrium state up to the machine precision. Moreover, one- and two-dimensional simulations are performed to test the ability of the current methods to capture small perturbation of such equilibrium state, and the genuine high order accuracy in smooth regions.

1. Introduction

Hydrodynamical evolution under gravitational fields frequently arises in many applications [1,2] including the astrophysics and the numerical weather prediction. Specific examples include the researches of atmospheric phenomena that are intrinsical in numerical weather prediction and in climate modelling. In addition, there are a wide variety of studies in astrophysics for instance modelling solar climate and simulating supernova explosions. In general, the hydrodynamical evolution can be modeled by the compressible Euler equations coupled with a gravitational source term:

$$\begin{aligned}
&\rho_t + \nabla\cdot(\rho\mathbf{u}) = 0,\\
&(\rho\mathbf{u})_t + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u} + p\mathbf{I}_d) = -\rho\nabla\phi,\\
&E_t + \nabla\cdot((E+p)\mathbf{u}) = -\rho\mathbf{u}\cdot\nabla\phi,
\end{aligned} \tag{1}$$

* Corresponding author.
E-mail address: gangli1978@163.com (G. Li).

https://doi.org/10.1016/j.amc.2018.01.059

where $\mathbf{x} \in \mathcal{R}^d$ $(d = 1, 2, 3)$ is the spatial variable, ρ denotes the fluid density, $\mathbf{u}$ is the velocity, p represents the pressure, and $E = \frac{1}{2}\rho\|\mathbf{u}\|^2 + \rho e$ (e is internal energy) is the non-gravitational total energy which includes the kinetic and internal energy of the fluid. $\phi = \phi(\mathbf{x})$ is the time independent gravitational potential. The operators ∇, $\nabla\cdot$ and $\otimes$ are the gradient, divergence and tensor product in $\mathcal{R}^d$, respectively, and $\mathbf{I}_d$ stands for the identity matrix. To close the system, the pressure p is linked to the density and the internal energy through an equation of state denoted by

$$p = (\gamma - 1)\rho e = (\gamma - 1)\big(E - \rho\|\mathbf{u}\|^2/2\big), \tag{2}$$

with γ being the ratio of specific heats.

The system (1) belongs to hyperbolic balance laws and admits steady state solutions (also called as hydrostatic equilibrium state), in which the source term is exactly balanced by the non-zero flux gradient. Specifically speaking, there are two well-known hydrostatic equilibriums states, i.e., the isothermal [3] and the isentropic equilibrium state [2], which will be explained in detail in Section 2. Many practical problems [2–9] involve nearly steady state flows under gravitational fields, therefore it is essential to correctly capture the effect of gravitational force in these simulations, especially if a long-time integration is involved, for example in the modeling galaxy formation [2] and in the atmosphere modeling [9]. However,the standard numerical methods generally fail to maintain the steady state exactly, and result in spurious numerical oscillations even with much refined mesh [10]. Greenberg et al. [11,12] in 1997 originally introduced well-balanced methods, which preserve exactly the steady state solutions up to the machine precision. In addition, compared with the non-well-balanced ones, the well-balanced methods can accurately resolve small perturbations of such steady state with relatively coarse meshes [10,13].

In recent years, well-balanced methods have attracted much attention. LeVeque and Bale [14] extended the quasi-steady wave-propagation methods to the Euler equations under a static gravitational fields. Finite volume well-balanced discretizations with respect to dominant hydrostatics have been proposed by Botta et al. [9] for the nearly hydrostatic flows in the numerical weather prediction. Xu and his collaborators [4–6] have extended the gas-kinetic scheme to the multidimensional gas dynamic equations. Käppeli and Mishra [2] have proposed well-balanced finite volume schemes for the isentropic hydrostatic equilibrium. High order well-balanced finite difference weighted essentially non-oscillatory (WENO) schemes for the isothermal equilibrium are introduced in [3,15] by means of the reformulation of the governing equations. The first attempt of discontinuous Galerkin (DG) methods the isothermal model has been conducted by Li and Xing [16] based on the technique in [3]. Recently, Li and Xing designed high order well-balanced finite volume WENO schemes for both isothermal and isentropic models [17]. More recently, Li and Xing developed high order well-balanced finite difference WENO schemes [18] and well-balanced DG methods [19] for the isentropic models. Well-balanced finite volume schemes for the general hydrostatic equilibrium without any assumption of a thermal equilibrium are recently studied in [1,2,8]. Other related work on well-balanced methods can be found in [20–23].

The main objective of this study is to develop high order well-balanced DG methods for the Euler equations at isentropic equilibrium state under gravitational fields. DG methods are a class of finite element methods using discontinuous piecewise polynomial space as the solution and test function spaces (see [24,25] for a brief historic review). Several advantages of the DG method, including its accuracy, easy implementation of parallel computing, flexibility for hp-adaptation, convenient treatment for the boundary conditions and arbitrary geometry and meshes, make it useful for a wide range of applications [26–28].

Herein, in order to achieve well-balanced property, we first reformulate the source term in an equivalent form by means of the hydrostatic equilibrium state. Then, we propose well-balanced numerical fluxes as well as a novel source term approximation. Ultimately, the high order numerical approximations to the fluxes gradient are exactly balanced with those to the gravitational source term for the isentropic equilibrium state.

This paper is organized as follows. In Section 2, we present well-balanced DG methods for the one-dimensional problems. Subsequently, we extend the proposed well-balanced methods to multi-dimensional problems in Section 3. Section 4 contains extensive one- and two-dimensional numerical results to demonstrate the performance of proposed DG methods. Some conclusions are given in Section 5.

2. Well-balanced DG methods for one-dimensional cases

In this section, we first present the mathematical model of one-dimensional cases as well as hydrostatic equilibrium states. Subsequently, we construct well-balanced DG methods.

2.1. The mathematical model

In one spatial dimension, the model takes the following form

$$\begin{aligned} &\rho_t + (\rho u)_x = 0, \\ &(\rho u)_t + (\rho u^2 + p)_x = -\rho\phi_x, \\ &E_t + ((E+p)u)_x = -\rho u\phi_x. \end{aligned} \tag{3}$$