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Commutator estimate and its application to regularity criteria of the dissipative quasi-geostrophic equation

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ABSTRACT

A new commutator estimate with respect to a nonlinear convection upper bounded by a single partial derivative component in Hilbert spaces is obtained. As an application, regularity criteria on the supercritical quasi-geostrophic equation are obtained provided that solution growth conditions are assumed to involve a single partial derivative component.

1. Introduction

The motion of three-dimensional incompressible viscous fluids is governed by the Navier–Stokes equations showing the balance of inertial force with pressure and viscous forces

$$\partial_t u + u\cdot\nabla u + \nabla p + \Delta u = 0 \quad \text{in } R^3. \tag{1.1}$$

Here the nonlinear convective acceleration $u\cdot\nabla u$ is coupled with the gradient pressure ∇p. This leads to the absence of maximal principle and thus the lack of suitable a priori estimates to solve Leray's open problem (see Leray [25]) on the existence of global regular solutions. A variety of methods have been developed in an effort to control the nonlinearity (see, for example, Serrin [26], Kato [22], Caffarelli et al. [2] and Tao [28]). For the understanding of the convective acceleration in relation with the viscous force, the two-dimensional dissipative quasi-geostrophic equation

$$\partial_t\theta + u\cdot\nabla\theta + (-\Delta)^\alpha\theta = 0 \quad \text{in } R^2 \qquad (u = (-\Delta)^{-1/2}(\partial_y\theta, -\partial_x\theta)) \tag{1.2}$$

was introduced (see Constantin et al. [8], Constantin and Wu [9] and Wu [29]) for $0\le\alpha\le 1$.

In the demonstration of the regularity properties of the Navier–Stokes flow via the energy method, it is necessary to provide a suitable estimate of the nonlinear term $(-\Delta)^s(u\cdot\nabla u)$ or the commutator

$$(-\Delta)^s(u\cdot\nabla u) - u\cdot\nabla(-\Delta)^s u, \tag{1.3}$$

since the term $u\cdot\nabla(-\Delta)^s u$ is orthogonal to $(-\Delta)^s u$ due to the incompressibility condition of the fluid motion. The commutator analysis with respect to the convective acceleration is originated from the estimate of Kato and Ponce [23] expressed as

$$\|J^s(u\cdot\nabla v) - u\cdot\nabla J^s v\|_{L_p} \le c(\|\nabla u\|_{L_\infty}\|J^{s-1}\nabla v\|_{L_p} + \|J^s u\|_{L_p}\|\nabla v\|_{L_\infty}), \tag{1.4}$$

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for $J=(1-\Delta)^{1/2}$, $s>0$ and $1<p<\infty$. This estimate was used by Ju [19] to obtain a Serrin-like regularity criterion on the quasi-geostrophic Eq. (1.2). Eq. (1.4) was combined with a technique of [5] to demonstrate a Serrin-type regularity criterion (see Dong and Chen [13]) of (1.2) with respect to the $B^0_{p,\infty}-L_q$ norm assumption

$$\int_0^T \|\nabla\theta(t)\|^q_{B^0_{p,\infty}}dt<\infty \quad \text{with} \quad \frac{1}{p}+\frac{\alpha}{q}=\alpha, \quad \frac{2}{\alpha}\le p\le\infty. \tag{1.5}$$

This regularity criterion was further extended by Dong and Pavlovic [15] under the $B^s_{p,\infty}-L_q$ norm assumption

$$\int_0^T \|\nabla\theta(t)\|^q_{B^s_{p,\infty}}dt<\infty \quad \text{with} \quad \frac{2}{p}+\frac{2\alpha}{q}=2\alpha+s-1, \tag{1.6}$$

after the use of a commutator estimate in Besov spaces [6] developed from [23].

There is a large literature on the quasi-geostrophic equation [1,4,9,12,14,18,20,27,30,31,35]. Especially, for critical case $\alpha=\frac{1}{2}$, the global existence of a general regular solution was obtained by Caffarelli and Vasseur [3] and Kiselev et al. [24] due to the validity of the maximum principle of (1.2). However, for the supercritical case $0<\alpha<\frac{1}{2}$, the global existence problem remains unsolved as the nonlinearity produced by the term $u\cdot\nabla\theta$ is difficult to be controlled by the dissipation described by the term $(-\Delta)^\alpha\theta$. Therefore, to approach to a global regular solution, regularity criteria stemmed from Serrin [26] are widely examined (see [10,11,16,32–34] and references therein).

The purpose of the present paper is to develop a commutator estimate which is applicable to a regularity criterion on the quasi-geostrophic equation. This analysis is partially motivated from the commutator estimate in Hilbert spaces obtained by Fefferman et al. [17]

$$\|(-\Delta)^s(u\cdot\nabla v)-u\cdot\nabla(-\Delta)^s v\|_{L_2}\le c\|J^{2s}\nabla u\|_{L_2}\|J^{2s}v\|_{L_2}, \tag{1.7}$$

where s is lower bounded by the dimension of the fluid domain. The right-hand side of (1.7) in the L_2 space is simplified from the classical Kato–Ponce commutator estimate in L_p and L_q spaces. This also simplifies the application to fluid motion equations. Recently, Chen and Chen [7] considered the controlling of the commutator by partial derivative components presented separately in Besov spaces and the applications to the quasi-geostrophic equation. In the present study, with the development of the analysis of [7,17], we show the following commutator estimate result in Hilbert spaces.

Theorem 1.1. *Let $\theta\in H^{s+1}(R^2)$, $u=(u_1,u_2)=(\partial_y\Lambda^{-1}\theta,-\partial_x\Lambda^{-1}\theta)$ with $\Lambda=(-\Delta)^{1/2}$, and*

$$\rho,\ \sigma\ge 0, \quad \rho+\sigma<1 \quad \text{and} \quad \rho+\sigma+s>2. \tag{1.8}$$

Then the following estimate

$$\|\Lambda^s(u\cdot\nabla\theta)-u\cdot\nabla\Lambda^s\theta\|_{L_2}\le c\|\Lambda^\rho\partial_x\theta\|_{H^{s-1}}\ \|\Lambda^\sigma\partial_y\theta\|_{H^{s-1}} \tag{1.9}$$

holds true.

The application of this commutator estimate gives rise to a new approach to the understanding of quasi-geostrophic Eq. (1.2). We will examine regularity criterion or global existence problem for the supercritical quasi-geostrophic Eq. (1.2) associated with the assumptions

$$\int_0^T \|\partial_x\theta\|^q_{H^{\beta-1}}dt<\infty \tag{1.10}$$

or

$$\int_0^T \frac{\|\partial_x\theta\|^q_{H^{\beta-1}}}{1+\ln\left(e+\|\nabla\theta\|^2_{L_p}\right)}dt<\infty \quad \text{with} \quad 1<p\le\frac{1}{\alpha} \tag{1.11}$$

for

$$1<q<2 \quad \text{and} \quad 2+\frac{2\alpha}{q}-2\alpha<\beta<3+\frac{2\alpha}{q}. \tag{1.12}$$

The regularity criterion with the growth condition on the complete gradient $\nabla\theta$ was examined by Chae [5]. The growth condition (1.10) involving the single partial derivative component $\partial_x\theta$ is comparable with that given by Yamazaki [34] in the following from

$$\int_0^T \|\partial_x\theta\|^q_{L_p}dt<\infty \quad \text{for } 4<\frac{2(1-\alpha)}{\alpha^2}<p<\infty \ \text{ and } \ \frac{1}{p}+\frac{\alpha}{q}\le\alpha. \tag{1.13}$$