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Solving uncertain heat equation via numerical method

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ABSTRACT

Uncertain heat equation is a type of uncertain partial differential equations driven by Liu processes. This paper proposes a concept of α-path for uncertain heat equation, and shows that the solution of an uncertain heat equation can be represented by a family of solutions of ordinary heat equations. And, a formula is derived to calculate expected value of solution of uncertain heat equation. Moreover, a numerical method is designed to solve uncertain heat equation. Several examples are given to illustrate the efficiency of the numerical method.

1. Introduction

As a new mathematical system to model human belief degrees, uncertainty theory was established by Liu [4] in 2007 and perfected by Liu [6] in 2009. It is based on normality, duality, subadditivity and product axioms. Uncertain process is a sequence of uncertain variables indexed by time to describe the dynamical behavior of uncertain phenomena. The origin of uncertain process was traced to the pioneering work of Liu [5] in 2008. As an uncertain counterpart of Wiener process, Liu process was developed by Liu [6] to deal with white noise. It is a Lipschitz continuous uncertain process with stationary and independent increments. Following that, uncertain calculus was built by Liu [6] to handle the integration and differentiation of functions of uncertain processes.

Uncertain differential equation driven by Liu process was first presented by Liu [5]. Under linear growth condition and Lipschitz continuous condition, Chen and Liu [1] proved a existence and uniqueness theorem for uncertain differential equations. And they obtained an analytic solution for a linear uncertain differential equation. Later on, Liu [10] and Yao [20] studied a spectrum of analytic methods to solve some special classes of nonlinear uncertain differential equations. Nevertheless, it is difficult to obtain the analytic solutions for general uncertain differential equations. Fortunately, Yao and Chen [19] proved that the solution of an uncertain differential equation can be represented by a spectrum of ordinary differential equations, which supplies a possibility to obtain numerical solutions for general uncertain differential equations. For solving uncertain differential equation, some numerical methods were designed among others by Yao and Chen [19], Yang and Shen [14], Yang and Ralescu [15], Wang et al. [12], and Gao [3]. At present, uncertain differential equation has been widely applied in many fields by many researchers, such as uncertain finance (Liu [6,8], Chen and Gao [2], Liu et al. [11]), and uncertain game (Yang and Gao [13,16]).

Uncertain partial differential equation driven by Liu process was first proposed by Yang and Yao [17]. They also studied uncertain heat equation whose heat source is often affected by the uncertain interference. And they obtained the solution and inverse uncertainty distribution of solution for a special class of uncertain heat equations. Later then, Yang and Ni

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[18] proved an existence and uniqueness theorem of solution for general uncertain heat equations under linear growth condition and Lipschitz condition.

However, it is difficult to find analytic solutions of general uncertain heat equations. Thus a numerical method has to been designed. This paper aims at giving a numerical method to solve uncertain heat equations. The rest of the paper is arranged as follows. Section 2 reviews some basic definitions and results in uncertainty theory. Section 3 introduces uncertain heat equation. Section 4 proposes the concept of α-path for uncertain heat equation. Section 5 proves a theorem that the solution of an uncertain heat equation can be represented by a family of solutions of ordinary heat equations. Section 6 shows that the inverse uncertainty distribution of solution of uncertain heat equation is just the α-path of uncertain heat equation. And a numerical method is designed to obtain the inverse uncertainty distribution. Section 7 gives a formula to calculate expected value of solution of uncertain heat equation and presents a numerical method to get the expected value of solution. Section 8 gives some examples to illustrate the efficiency of this numerical method. At last, Section 9 gives a brief summary.

2. Preliminaries

In this section, we introduce some fundamental concepts in uncertainty theory including uncertain variable, uncertain process and uncertain field.

Definition 2.1. (Liu [4]) Let $\mathcal{L}$ be a σ-algebra on a nonempty set Γ. A set function $\mathcal{M}: \mathcal{L} \to [0, 1]$ is called an uncertain measure if it satisfies the following axioms,

Axiom 1. (Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ;

Axiom 2. (Duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ;

Axiom 3. (Subadditivity Axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \le \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

Besides, in order to provide the operational law, Liu [6] defined the product uncertain measure on the product σ-algebre $\mathcal{L}$ as follows.

Axiom 4. (Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$. The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \ldots$, respectively.

Theorem 2.1. *(Liu [7], Monotonicity Theorem) Uncertain measure $\mathcal{M}$ is a monotone increasing set function. That is, for any events $\Lambda_1 \subset \Lambda_2$, we have*

$$\mathcal{M}\{\Lambda_1\} \le \mathcal{M}\{\Lambda_2\}.$$

An uncertain variable is a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers. Uncertainty distribution Φ: $\Re \to [0, 1]$ of an uncertain variable ξ is defined as $\Phi(x) = \mathcal{M}\{\xi \le x\}$. And an uncertainty distribution $\Phi(x)$ is said to be regular if it is a continuous and strictly increasing function with respect to x at which $0 < \Phi(x) < 1$, and

$$\lim_{x \to -\infty} \Phi(x) = 0, \ \lim_{x \to +\infty} \Phi(x) = 1.$$

If ξ is an uncertain variable with regular uncertainty distribution $\Phi(x)$, then we call inverse function $\Phi^{-1}(\alpha)$ as the inverse uncertainty distribution of ξ. The uncertain variables $\xi_1, \xi_2, \ldots, \xi_m$ are said to be independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^{m} \{\xi_i \in B_i\}\right\} = \bigwedge_{i=1}^{m} \mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \ldots, B_m$ of real numbers.

The operational law of uncertain variables was proposed by Liu [7] to calculate the inverse uncertainty distribution of strictly monotonous function as the following theorem.

Theorem 2.2. *(Liu [7]) Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If the function $f(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with $x_{m+1}, x_{m+2}, \ldots, x_n$, then the uncertain variable*

$$\xi = f(\xi_1, \ldots, \xi_m, \xi_{m+1}, \ldots, \xi_n)$$

is an uncertain variable with inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)).$$