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Lie symmetry analysis of a class of time fractional nonlinear evolution systems

Khongorzul Dorjgotov[a], Hiroyuki Ochiai[b], Uuganbayar Zunderiya[c,*]

[a] Graduate School of Mathematics, Kyushu University, 744 Motooka, Fukuoka 819-0395, Japan
[b] Institute of Mathematics for Industry, Kyushu University, 744 Motooka, Fukuoka 819-0395, Japan
[c] Department of Mathematics, National University of Mongolia, P.B. 507/38, Chingiltei 6, Ulaanbaatar 15141, Mongolia

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ABSTRACT

We study a class of nonlinear evolution systems of time fractional partial differential equations using Lie symmetry analysis. We obtain not only infinitesimal symmetries but also a complete group classification and a classification of group invariant solutions of this class of systems. We find that the class of systems of differential equations studied is naturally divided into two cases on the basis of the type of a function that they contain. In each case, the dimension of the Lie algebra generated by the infinitesimal symmetries is greater than 2, and for this reason we present the structures and one-dimensional optimal systems of these Lie algebras. The reduced systems corresponding to the optimal systems are also obtained. Explicit group invariant solutions are found for particular cases.

1. Introduction

Lie group analysis provides an efficient algorithmic approach for studying the symmetry of ordinary and partial differential equations and for solving such equations [1–8]. Recently, the methods of symmetry analysis have been extended to solving fractional partial differential equations (FPDE) [9–15] and systems thereof [16–18]. In recent years, there has been growing interest in FPDEs in fields of both pure and applied mathematics. In particular, FPDEs have been studied in the contexts of fractals, acoustics, control theory and signal processing.

In this article, we consider the class of time fractional nonlinear systems of the following form:

$$\begin{cases} \dfrac{\partial^\alpha u}{\partial t^\alpha} = v_x, \\ \dfrac{\partial^\alpha v}{\partial t^\alpha} = b^2(u)u_x, \end{cases} \tag{1}$$

where α is a positive non-integer number and $b(u)$ is a non-constant, sufficiently differentiable function. Here, fractional differentiation is defined in the Riemann–Liouville manner:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} := \begin{cases} \dfrac{\partial^n u}{\partial t^n}, & \text{for } \alpha \in \mathbb{N}, \\ \dfrac{1}{\Gamma(n-\alpha)} \dfrac{\partial^n}{\partial t^n} \displaystyle\int_0^t \frac{u(x,s)}{(t-s)^{\alpha-n+1}} ds, & \text{for } \alpha \in (n-1,n), \ \text{with } n \in \mathbb{N}. \end{cases} \tag{2}$$

* Corresponding author.
E-mail address: uuganbayar@smcs.num.edu.mn (U. Zunderiya).

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In [16], the class of time fractional linear evolution systems

$$\begin{cases} \dfrac{\partial^\alpha u}{\partial t^\alpha} = C^2(x) v_x, \\ \dfrac{\partial^\alpha v}{\partial t^\alpha} = u_x \end{cases} \tag{3}$$

where $C(x)$ is a sufficiently differentiable function and α is a positive non-integer number, is investigated using Lie symmetry analysis. Also, in [17], the nonlinear model of stationary transonic plane-parallel gas flows

$$\begin{cases} \dfrac{\partial^\alpha u}{\partial t^\alpha} = v_x, \\ \dfrac{\partial^\alpha v}{\partial t^\alpha} = -u u_x, \end{cases} \tag{4}$$

with $0 < \alpha < 1$, was studied using Lie symmetry analysis. The Lie symmetries, some reduced systems of ODEs and some partial solutions of system (4) are obtained in [17]. Substituting $\bar{u}(x,t) = -u(x,t)$ and $\bar{v}(x,t) = -v(x,t)$ into (4), we obtain the following equivalent fractional system:

$$\begin{cases} \dfrac{\partial^\alpha \bar{u}}{\partial t^\alpha} = \bar{v}_x, \\ \dfrac{\partial^\alpha \bar{v}}{\partial t^\alpha} = \bar{u}\bar{u}_x. \end{cases}$$

This corresponds to the particular case $b(u) = \sqrt{u}$ for the system given in (1). It is thus seen that (1) can be viewed as a nonlinear version of (3) and a generalization of (4) with respect to the above-mentioned substitution. Hence, the results of this paper generalize the results of [17].

The importance of finding exact solutions of (1) lies in the fact that if $(u(x,t), v(x,t))$ solves (1), then $u(x, t)$ solves the sequential equation

$$\frac{\partial^\alpha}{\partial t^\alpha}\frac{\partial^\alpha}{\partial t^\alpha} u = \left(b^2(u) u_x\right)_x. \tag{5}$$

For example, in the case $\alpha = 1$, Eq. (5) becomes the well-known nonlinear wave equation and in the case $\alpha = \frac{1}{2}$, the component $u(x, t)$ of the solutions of (1) with $t > 0$ is a solution to the nonlinear heat equation with source

$$u_t = \left(b^2(u) u_x\right)_x + \frac{1}{\sqrt{\pi}}\frac{g(x)}{\sqrt{t}}, \quad g(x) = \frac{1}{\sqrt{\pi}}\int_0^t \frac{u(x,\tau)}{\sqrt{t-\tau}} d\tau\Big|_{t=0}, \quad t > 0,$$

by virtue of the formula [20]

$$\frac{\partial^p}{\partial t^p}\frac{\partial^q}{\partial t^q} f(x,t) = \frac{\partial^{p+q} f(x,t)}{\partial t^{p+q}} - \sum_{j=1}^{n}\left[\frac{\partial^{q-j} f(x,t)}{\partial t^{q-j}}\right]\Big|_{t=0} \frac{t^{m-p-j}}{\Gamma(1+m-p-j)}, \text{ with } m-1 \le p < m, \quad n-1 \le q < n.$$

However, it should be noted that, in general, the Lie group of point transformations that leaves the system in (1) invariant, does not necessarily correspond to a Lie group of point transformations that leaves the single equation in (5) invariant.

We study the system given in (1) using Lie symmetry analysis. More explicitly, we present a complete group classification depending on the function $b(u)$ and describe the structure of Lie algebras generated by the infinitesimal symmetries of (1). After obtaining the group classification of (1), we proceed to finding optimal systems of Lie algebras and the reduced systems of ODEs. Using these optimal systems, we also classify the group invariant solutions corresponding to the infinitesimal symmetries for $0 < \alpha < 1$.

The organization of this paper is as follows. In Section 2, we present a simple introduction to the Lie symmetry analysis of systems of FPDEs and provide formulas useful in studying (1). In Section 3, we carry out a complete group classification with respect to the function $b(u)$. In Section 4, we further investigate the structure of the corresponding Lie algebras of infinitesimal symmetries and determine the optimal systems. We also reduce (1) to systems of fractional and non-fractional ODEs in accordance with these optimal systems. For particular cases, some explicit solutions are given in Section 5.

2. Lie symmetry analysis for a system of fractional partial differential equations

To begin, we present the basic definitions and formulas needed to carry out the Lie symmetry analysis of a system of FPDEs. The general form of a system of time fractional PDEs with two independent variables x and t is as follows:

$$\begin{cases} \dfrac{\partial^\alpha u(x,t)}{\partial t^\alpha} = F_1(x,t,u,u_x,u_{xx},\ldots,v,v_x,v_{xx},\ldots), \\ \dfrac{\partial^\alpha v(x,t)}{\partial t^\alpha} = F_2(x,t,u,u_x,u_{xx},\ldots,v,v_x,v_{xx},\ldots), \end{cases} \tag{6}$$