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Semi-implicit second order schemes for numerical solution of level set advection equation on Cartesian grids ☆

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ABSTRACT

A new parametric class of semi-implicit numerical schemes for a level set advection equation on Cartesian grids is derived and analyzed. An accuracy and a stability study is provided for a linear advection equation with a variable velocity using partial Lax–Wendroff procedure and numerical von Neumann stability analysis. The obtained semi-implicit κ-scheme is 2nd order accurate in space and time in any dimensional case when using a dimension by dimension extension of the one-dimensional scheme that is not the case for analogous fully explicit or fully implicit κ-schemes. A further improvement is obtained by using so-called Corner Transport Upwind extension in two-dimensional case. The extended semi-implicit κ-scheme with a specific (velocity dependent) value of κ is 3rd order accurate in space and time for a constant advection velocity, and it is unconditional stable according to the numerical von Neumann stability analysis for the linear advection equation in general.

1. Introduction

In this work, we derive a new class of semi-implicit 2nd order schemes for numerical solutions of a representative linear advection equation

$$\partial_t u(x,t) + \vec{V} \cdot \nabla u(x,t) = 0, \quad u(x,0) = u^0(x)$$

with a variable velocity $\vec{V} = \vec{V}(x)$. We are interested in level set methods [34,39] when this equation is used to track implicitly given interfaces, and when discontinuous profiles in the solution are not expected in general. The implicit tracking of interfaces can be found in any front propagation problems solved by level set methods, see, e.g., [34,39] and the references there. A typical application is a two-phase flow of immiscible fluids where an interface between the phases must be tracked to distinguish the different physical properties of fluids [7,9,16,19,21,41,46,50]. Furthermore we mention a tracking of fire front in forests [2,14], and a tracking of water table for groundwater flows [8,18].

We consider Cartesian grids that are often applied in the context of level set methods [14,15,34,39,41]. We consider the linear advection equation on Cartesian grids also as a starting point for a study of more complex equations like a nonlinear advection equation for a motion in normal direction [12,14,30,35,39] and computations on unstructured grids [9,12,17]. We

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are interested here in Eulerian type of numerical schemes of a finite difference form when a stencil of the scheme does not move in time like in Lagrangian type of numerical schemes [5,11]. Furthermore we restrict ourselves to the schemes using an implicit or a semi-implicit time discretization with a purpose of favorable stability properties when compared to the schemes of Eulerian type using a fully explicit time discretization.

The fully explicit schemes are standard numerical tool in level set methods for the solution of the linear advection equation [12,16,19,21,22,33,34,36,37,39,41,49]. The main advantage is their simplicity as the numerical solution, once the scheme is constructed, can be obtained directly without solving any algebraic system. On the other hand the well-known restriction of fully explicit schemes with fixed stencils is a CFL stability condition on the choice of time steps that depends, among other, on a length of grid steps.

Although the CFL restriction is not considered as a disadvantage in general, it can be critical for applications with irregular computational domains for which the boundaries are treated implicitly like in Cartesian cut cell methods [25], immersed interface methods [14,27,28,49], ghost fluid methods [7,29] and similar. In the quoted methods the presence of arbitrary small cut cells can give locally an arbitrary small grid size that results in an unrealistic CFL restriction if no modifications of the numerical scheme is provided.

Recently some publications [14,30,32] have been dealing with semi-implicit finite volume schemes for a general advection equation. The main idea is that the implicit time discretization is used only for the values of numerical solution at inflow boundaries of computational cells. The semi-implicit schemes can be advantageous when solving the advection equation on implicitly given computational domain as it appears, e.g., when constructing a so-called “extension” velocity in level set methods, see [1,52]. This approach is used in [14] where the linear advection equation is solved by a particular semi-implicit method on a time dependent domain given by positions of a fire front in a forest, and where no cut-cell problem occurs in numerical simulations.

Although some analysis is provided in [14,30,32] the particular semi-implicit schemes are derived ad hoc. In this work we present a unified representation of such semi-implicit schemes using a novel approach of partial Lax–Wendroff procedure and study their accuracy and stability properties. The Lax–Wendroff [23] (or Cauchy–Kowalevski [42]) procedure in its full form replaces the time derivatives of the solution in Taylor series by the space derivatives [26,42]. This procedure is used in a derivation of high order ADER (Arbitrary DERivatives) schemes that are applied to a variety of applications, see, e.g., [42] and the references there. In our approach we apply the steps of Lax–Wendroff procedure only partially by allowing the mixed time-space derivatives of the solution in Taylor series.

We use this procedure with an approach of fully explicit κ-scheme [44,45,47] that includes as particular cases some popular numerical schemes like Lax–Wendroff and Fromm scheme [26,47] or QUICKEST scheme [24,47]. The general formulation of the semi-implicit κ-scheme gives us an opportunity to use special choices of the parameter κ to improve the accuracy and the stability of the scheme in special cases, and to adapt the scheme near boundaries.

To show some advantages of the partial Lax–Wendroff procedure with respect to the full procedure, we compare the semi-implicit κ-scheme with an analogous fully implicit κ-scheme derived in this paper using the full Lax–Wendroff procedure. We study the stability conditions of all presented schemes using von Neumann stability analysis [20,43,47] realized in a numerical way as suggested in [3,4].

The semi-implicit κ-scheme is unconditionally stable in the one-dimensional case for all relevant values of κ that is not the case for the fully implicit κ-scheme. We show that this property can be used for the immersed interface methods when boundary conditions are defined on an implicitly given boundary of computational domain. Furthermore we derive a novel particular variant of the semi-implicit κ-scheme by defining a variable (velocity dependent) value of the parameter κ. The scheme is 3rd order accurate in space and time for a constant velocity in 1D.

Opposite to the fully implicit κ-scheme (and also the fully explicit κ-scheme), the semi-implicit κ-scheme remains 2nd order accurate in space and time in several dimensions when using a standard dimension by dimension extension of 1D scheme on Cartesian grids. Unfortunately, this extension of the semi-implicit κ-scheme in several dimensions is conditionally stable in general.

To improve the stability of two-dimensional semi-implicit κ-scheme we apply the idea of Corner Transport Upwind (CTU) extension [6,26] by adding an additional discretization term to the scheme. The main result is a novel scheme with the velocity dependent value of κ using the CTU extension that is unconditionally stable according to the numerical von Neumann stability analysis. Moreover the scheme is 3rd order accurate in the case of constant velocity. For several representative numerical experiments this variant of the semi-implicit κ-scheme gives the most accurate results among other considered choices of κ.

The paper is organized as follows. In Section 2 we begin with the one-dimensional case where the fully implicit and the semi-implicit κ-schemes are derived. In Section 3 we discuss the properties of semi-implicit κ-scheme in several dimensions when obtained by the dimension by dimension extension. Furthermore the Corner Transport Upwind extension of the scheme and the treatment of boundary conditions on implicitly given boundary are described. In Section 4 several numerical experiments are presented that involve examples on an implicitly given computational domain, an example with largely varying velocity, and two standard benchmark examples for tracking of interfaces. Finally we conclude the results in Section 5.