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Solving singularly perturbed problems by a weak-form integral equation with exponential trial functions

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ABSTRACT

The second-order singularly perturbed problem is transformed to a singularly perturbed parabolic type partial differential equation by using a fictitious time technique. Then we use Green's second identity to derive a boundary integral equation in terms of the adjoint Trefftz test functions, namely a weak-form integral equation method (WFIEM). It accompanying with the exponential trial functions, which are designed to satisfy the boundary conditions automatically, can provide very accurate numerical solutions of linear and nonlinear singularly perturbed problems. For the latter problem the iterative procedure is convergent very fast.

1. Introduction

In this paper, we propose a new method to solve the second-order linear and nonlinear ordinary differential equations (ODEs), whose highest order derivative term is multiplied by a small perturbing parameter. When the boundary conditions are imposed to the ODEs, the resulting problems are singularly perturbed boundary value problems (SPBVPs), which exhibit boundary layer behavior with the solution varying rapidly in a narrow region.

For the SPBVPs there have been many asymptotic approximation methods, and the numerical methods based on these concepts to match the boundary layer behavior were well developed, which are uniformly valid with respect to the perturbation parameter [1]. In addition to the asymptotic approximation methods, many computational methods have been developed for solving the SPBVPs, to name a few, Bender and Orszag [2], Shampine and Gear [3], Nayfeh [4], Kevorkian and Cole [5,6], O'Malley [7], and De Jager and Jiang [8]. The survey was carried out by Kadalbajoo and Patidar [9] and Kadalbajoo and Gupta [10]. The readers may refer [11–19] for recently developed numerical methods to solve the SPBVPs.

The arrangement of this paper is given as follows. In Section 2, we propose a variable transformation method, such that the linear singularly perturbed problem becomes a singular parabolic problem. Then, in Section 3 we employ Green's second identity to derive a boundary integral equation for finding the singular solution in terms of the adjoint eigenfunctions derived in Section 4. In Section 5, we develop a weak-form integral equation method for the constant coefficients linear singularly perturbed problem. In Section 6, we develop a weak-form integral equation method for the time-varying coefficients linear singularly perturbed problems. A numerical algorithm based on the resultant weak-form integral equation method

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(WFIEM) is developed in Section 7, where we express the trial solution in terms of exponential trial functions as bases, which are designed to satisfy the boundary conditions automatically. The numerical examples are given in Section 8 to validate the effectiveness of the present method. In Section 9, we introduce a new iterative scheme to solve the nonlinear singularly perturbed problems and numerical examples are given. Finally, some conclusions are drawn in Section 10.

2. Transformation of ODE to PDE

Let us consider a second-order linear singularly perturbed ODE, describing a boundary layer problem of singularly perturbed boundary value problem (SPBVP) under a time varying loading:

$$\varepsilon\ddot{x}(t) + b\dot{x}(t) + cx(t) = f(t), \quad 0 < t < t_f, \tag{1}$$

$$x(0) = \alpha, \quad x(t_f) = \beta, \tag{2}$$

where ε is a small parameter, and b and c are time-dependent functions.

We first treat the SPBVP that b and c are constants. Then, in Section 6 we will consider problem (1), when b and c are time-dependent functions. As that done by Liu [20], let

$$u(t,\tau) = (1+\tau)x(t), \tag{3}$$

where τ is a fictitious time variable. It follows that

$$\begin{aligned} u_\tau(t,\tau) &= x(t), \\ u_t(t,\tau) &= (1+\tau)\dot{x}(t), \\ u_{tt}(t,\tau) &= (1+\tau)\ddot{x}(t), \end{aligned} \tag{4}$$

where the subscript denotes the partial derivative. Then, upon multiplying Eq. (1) by $1+\tau$ and using Eqs. (3) and (4), we can obtain

$$\varepsilon u_{tt}(t,\tau) + bu_t(t,\tau) + cu(t,\tau) = (1+\tau)f(t). \tag{5}$$

When we add a term u_τ on the right-hand side of Eq. (5), we also need to add a term $u/(1+\tau)$ on the left-hand side to balance the equality, such that according to Eqs. (3) and (4), we have the following linear parabolic type partial differential equation (PDE) under boundary conditions and initial condition:

$$u_\tau(t,\tau) = \varepsilon u_{tt}(t,\tau) + bu_t(t,\tau) + cu(t,\tau) + \frac{u(t,\tau)}{1+\tau} - (1+\tau)f(t), \tag{6}$$

$$u(0,\tau) = (1+\tau)x(0) = \alpha(1+\tau), \tag{7}$$

$$u(t_f,\tau) = (1+\tau)x(t_f) = \beta(1+\tau), \tag{8}$$

$$u(t,0) = x(t), \tag{9}$$

where Eq. (9) is a direct result of Eq. (3) by inserting $\tau = 0$. Eq. (6) is itself a singularly perturbed convective PDE, because ε is a small parameter.

3. Boundary integral equation method

We first derive a weak-form integral equation for Eq. (1) when b and c are constants. For the case of time-varying coefficients $b(t)$ and $c(t)$ in Eq. (1), a weak-form integral equation will be derived in Section 6. In order to develop the new method we need the following results.

Theorem 1 (Green's second identity). *Let Ω be a bounded region in the plane (t,τ) with a counter-clockwise contour Γ consists of finitely many smooth curves. Let $u(t,\tau)$ and $v(t,\tau)$ be functions that are differentiable in Ω and continuous on $\bar{\Omega}$. Then,*

$$\int\int_\Omega (v\mathcal{H}u - u\mathcal{H}^*v)dtd\tau = \oint_\Gamma [\varepsilon(uv_t - vu_t) - buv]d\tau - uvdt, \tag{10}$$

where

$$\mathcal{H}u := u_\tau - \varepsilon u_{tt} - bu_t - cu - \frac{u}{1+\tau}, \tag{11}$$

$$\mathcal{H}^*v := -v_\tau - \varepsilon v_{tt} + bv_t - cv - \frac{v}{1+\tau} \tag{12}$$

are, respectively, the linear parabolic operator and the adjoint linear parabolic operator.