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Virtual element method for two-dimensional linear elasticity problem in mixed weakly symmetric formulation

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ABSTRACT

We propose and analyze a virtual element method for two-dimensional linear elasticity problem in mixed weakly symmetric formulation, that is to say, stresses are not require to be symmetric, but only to satisfy a weaker condition based on Lagrange multipliers. The proposed method is well-posed, and the error bounds are shown to be uniform with the incompressibility parameter λ. Numerical tests confirm the convergence rate that is expected from the theory.

1. Introduction

Recently, a new discretization approach called the Virtual Element Method (VEM) has been developed. The VEM was first introduced in [1], as an evolution of modern Mimetic Finite Difference method (MFD) [2,3], belonging to a family of methods that allow the use of general polygonal and polyhedral meshes. It is worth mentioning that making use of polygonal meshes brings a range of advantages. This include, for instance, the greater robustness to mesh distortion, more efficient and easier adaptivity, greater flexibility in the mesh generation process (see for instance [2] or [4] for more details). Of course, there are a variety of methods that can deal with polygonal or polyhedral meshes such as the Hybrid Discontinuous Galerkin (HDG) [5], the Hybrid High-Order (HHO) methods [6], the Polygonal Finite Elements methods [7],the Weak Galerkin (WG) methods [8], and the Mimetic Finite Difference method (MFD) [2,3]. The construction of virtual element are very similar to the classical finite element, for example, an appropriate set of degrees of freedom and shape functions are defined to satisfy unisolvence, while in virtual element context, the shape functions are no longer polynomials. This feature allows us to design a family of conforming elements like H^{α}-conforming (α is a positive integer) [9], $H(div)$-conforming and $H(curl)$-conforming elements [10] on complicated element geometries. As a consequence, VEM is not restrict to low-order converge and even can be easily applied to three dimensions and allowed to use nonconvex elements. Another feature, also key point of the VEM is to construct computable bilinear forms that can keep certain accuracy only using the degrees of freedom, as the non-polynomial shape functions will be considered. After designing suitable discrete bilinear forms, VEM is able to make use of very general polygonal/polyhedral meshes without the need to integrate complex non-polynomial functions on the elements and without loss of accuracy. The VEM has been developed successfully in a wide range of problems: the linear elasticity problems, both for the two-dimensional case and the three-dimensional case [11,12], a stream formulation of VEM for the Stokes problem [13], divergence free virtual elements for the Stokes problem [14], the non-linear elastic and inelastic deformation problems mainly focusing on a small deformation regime [15], the Darcy problem in mixed form [16], the plate bending problem [17], the Steklov eigenvalue problem [18], the general second order elliptic problems

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in primal [19] and mixed form [20], the advection–diffusion problems [21], the Cahn–Hilliard equation [22], the Helmholtz problem [23], the discrete fracture network simulations [24].

The main aim of this paper is to develop the VEM for the elasticity problem in mixed formulation, which adopts stresses and displacements as unknowns. In general, there are two approaches to design finite element methods for linear elasticity. The first approach is to enforce the symmetry of the stress tensor exactly [25–31]. In this category, the VEM was considered in [31]. In this paper, we will consider the second approach that is to enforce the symmetry weakly. In the context of classical mixed finite element theory, many kinds of stable and converging finite element spaces have been thoroughly investigated by many works such as [32–36]. Classical mixed methods are indeed an appealing technique for the numerical solution of elasticity problems. They ensure the equilibrium condition and they make the constitutive law more explicit (see [36] for more details). It seems that the VEM for elasticity problem in mixed form is straightforward, considering that mixed formulation for elasticity is very similar to the diffusion problem, and it has been shown that the VEM is very suitable for the diffusion problem (see [16]), and there are many kinds of $H(div)$-conforming spaces that are BDM-like (Brezzi–Douglas–Marini) or RT-like (Raviart-Thomas) virtual element spaces, which have been designed, for instance, in [10,20]. However, this is not the case. For one thing, the stress tensor becomes the main variable but the symmetry of this tensor makes the construction of suitable finite element spaces much more complicated, when it comes to virtual element spaces, it seems to be much harder. For another, the construction of computable discrete bilinear form becomes more complicated because of the lack of uniform coercivity of the main bilinear form of the problem. Recently, the authors in [31] have established a symmetry stress/displacement VEM, which is a low-order scheme for plane elasticity problems. Our motivation is how to construct an arbitrary order virtual element method for elasticity problems in stress/displacement formulation. In [36], the authors present special ways of introducing mixed finite elements with reduced symmetry. By combing the ideas with VEM, we observed that it is possible to construct a virtual element method for elasticity problems in mixed weakly symmetric formulation.

In the present paper, we develop the ideas in [36] to virtual element framework and propose a virtual element method that is suitable for linear elasticity problems in mixed weakly symmetric formulation. By our theoretical analysis, the proposed method is well-posed, and the error bounds are shown to be uniform with the incompressibility parameter λ. Numerical tests verifies the analysis and shows that the proposed method is also stable with respect to the shape of the mesh polygons.

The paper is arranged as follows. In section 2, we provide necessary notations. In Section 3, we simply introduce linear elasticity problem in mixed weakly symmetric formulation. In Section 4, we describe proposed virtual element method, and give a series of assumptions that can ensure the well posedness and convergence of discrete problem. In Section 5, we construct discrete spaces and discrete bilinear form that satisfy our assumptions. In addition, error estimates are also analysed in Section 5. Finally, we do some numerical tests in Section 6.

Throughout this paper, we use C, α^* and α_* to denote a positive constant independent of h and λ (but may depend on μ and dimension n), not necessarily the same at each occurrences.

2. Basic notations

Throughout the paper, $L^p(\mathcal{O})$ denotes the standard space of Lebesgue integrable functions over the domain $\mathcal{O}$, with the case $p = \infty$ allowed. We will use $|\cdot|_{s,\mathcal{O}}$ and $\|\cdot\|_{s,\mathcal{O}}$ to denote seminorm and norm, respectively, in the Sobolev space $H^s(\mathcal{O})$, while $(\cdot,\cdot)_{0,\mathcal{O}}$ will denote the $L^2(\mathcal{O})$ inner product. Often the subscript will be omitted when $\mathcal{O}$ is computational domain Ω. For k a non-negative integer, $\mathbb{P}_k(\mathcal{O})$ will denote the space of polynomials of degree $\leq k$ on $\mathcal{O}$. The symbol $H(\text{div};\mathcal{O})$ denotes the row vector fields in $[L^2(\mathcal{O})]^n$ whose divergence is in $L^2(\mathcal{O})$. When it comes to tensor fields, we define, in n dimensions,

$$\underline{\underline{H}}(d\underline{iv};\mathcal{O}) := \{\underline{\underline{\tau}} | \underline{\underline{\tau}} \in [L^2(\mathcal{O})]^{n\times n}, \quad d\underline{iv}\underline{\underline{\tau}} \in [L^2(\mathcal{O})]^n\}, \tag{2.1}$$

with its standard norm $\|\cdot\|^2_{\boldsymbol{H}(d\underline{iv};\mathcal{O})} = \|\cdot\|^2_{0,\mathcal{O}} + \|d\underline{iv}\cdot\|^2_{0,\mathcal{O}}$

$$\underline{\underline{H}}(d\underline{iv};\mathcal{O})_S := \{\underline{\underline{\tau}} | \underline{\underline{\tau}} \in \underline{\underline{H}}(d\underline{iv};\mathcal{O}), \quad \tau_{ij} = \tau_{ji}, \quad \forall i,j = 1,\dots,n\}, \tag{2.2}$$

$$\boldsymbol{\Sigma} := \underline{\underline{H}}(d\underline{iv};\Omega), \quad \boldsymbol{\Sigma}_S := \underline{\underline{H}}(d\underline{iv};\Omega)_S, \quad \boldsymbol{U} := [L^2(\Omega)]^n. \tag{2.3}$$

Remark 1. We require that $\underline{\underline{H}}(d\underline{iv};\mathcal{O})$ is obtained by patching row vectors. For example, in two dimensions, $\forall \underline{\underline{\tau}} \in \underline{\underline{H}}(d\underline{iv};\mathcal{O})$, $\underline{\underline{\tau}}$ can be written as $\underline{\underline{\tau}} = \begin{pmatrix} \underline{\tau_1} \\ \underline{\tau_2} \end{pmatrix}$, where $\underline{\tau_1} = \begin{pmatrix} \tau_{11} & \tau_{12} \end{pmatrix}$, $\underline{\tau_2} = \begin{pmatrix} \tau_{21} & \tau_{22} \end{pmatrix} \in H(\text{div};\mathcal{O})$. From above notations $d\underline{iv}\underline{\underline{\tau}}$ is nothing but $\begin{pmatrix} \text{div}\underline{\tau_1} \\ \text{div}\underline{\tau_2} \end{pmatrix}$.

When deal with tensor, it is convenient to introduce the following notations. Given a second order tensor $\underline{\underline{\tau}}$, we define its skew-symmetric part as

$$\underline{\underline{as}}(\underline{\underline{\tau}}) = \frac{1}{2}\left\{\underline{\underline{\tau}} - \underline{\underline{\tau}}^t\right\} \tag{2.4}$$