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An adaptive algorithm for TV-based model of three norms $L_q\left(q=\frac{1}{2},1,2\right)$ in image restoration

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ABSTRACT

In this paper, we present an adaptive method for the TV-based model of three norms $L_q\left(q=\frac{1}{2},1,2\right)$ for the image restoration problem. The algorithm with the L_2 norm is used in the smooth regions, where the value of $|\nabla u|$ is small. The algorithm with the $L_{\frac{1}{2}}$ norm is applied for the jumps, where the value of $|\nabla u|$ is large. When the value of $|\nabla u|$ is moderate, the algorithm with the L_1 norm is employed. Thus, the three algorithms are applied for different regions of a given image such that the advantages of each algorithm are adopted. The numerical experiments demonstrate that our adaptive algorithm can not only keep the original edge and original detailed information but also weaken the staircase phenomenon in the restored images. Specifically, in contrast to the L^1 norm as in the Rudin–Osher–Fatemi model, the L_2 norm yields better results in the smooth and flat regions, and the $L_{\frac{1}{2}}$ norm is more suitable in regions with strong discontinuities. Therefore, our adaptive algorithm is efficient and robust even for images with large noises.

1. Introduction

Image restoration is a significant problem in image processing. It has made many achievements in various fields so far, such as medical imaging analysis, communications and so on. Image denoising and deblurring is a fundamental research problem in image restoration. Mathematically, the image restoration is to recover the real image u from an observed noisy and blurred image z. The degraded process can be modeled as

$$z=Ku+n, \tag{1.1}$$

where K is a known linear blurring operator and n is an additive noise. It is a typical example of inverse problem and also an ill-posed problem. Recently, many varieties of methods and models have been utilized to regularize this ill-posed inverse problem. One of the important models is the total variation based restoration model proposed by Rudin, Osher and Fatemi (ROF) in [20]. In this model, the total variation of u is used as a regularization penalty functional. The corresponding image restoration can then be formulated as the following minimization problem:

$$\min_u\left(\mu\int_\Omega|\nabla u|dxdy+\frac{1}{2}\|Ku-z\|_2^2\right). \tag{1.2}$$

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Here $\mu > 0$ is a penalty parameter.

Several numerical methods have been proposed to solve the minimization problem (1.2). For example, in [6], [17], the authors proposed fast algorithms to solve the ROF model for denoising problems and provided rigorous proofs for the convergence of their algorithms.

The methods based on the Bregman iterative regularization were presented by many authors, such as linearized Bregman methods in [1,2,4,19], and split Bregman methods in [3,13,14,16,21]. The augmented Lagrangian method was proposed by J. Eckstein in optimization (see [15]) and was applied in the image processing.

The main idea of linearized Bregman methods is to deal with nonlinear terms in functional minimizations. The idea of operator splitting is used in the split Bregman method and the augmented Lagrangian method. Their convergence has been proved in [3,21]. These methods are significant in image restoration and image segmentation, because they are simple, fast, and efficient.

As we all know, the remarkable feature of the ROF model is to its ability of maintaining the sharp edges in images. However, a main drawback of the model is the appearance of the so-called staircase phenomenon. Hence, many researchers proposed several improvements.

The Tikhonov's regularization of using the L_2 norm was proposed in solving ill-posed problems (see [22]) and was also applied for the image restoration. Some models of using $L_q(1 < q < 2)$ norm were considered.

In the papers [5,11,12,18,23,24,26], $L_q(0 < q < 1)$ regularization was applied in the image processing. The $L_q(0 < q < 1)$ regularization has the following form:

$$\min_u \left(\mu\|Fu\|_q^q + \frac{1}{2}\|Ku - z\|_2^2\right), \tag{1.3}$$

where $0 < q < 1$, $\|Fx\|_q^q$ is the L_q quasi-norm in $\mathbb{R}^N$ and the operator F denotes the gradient ∇ or the identity I.

However, the $L_q(0 < q < 1)$ regularization leads to a non-convex and non-Lipschitz optimization problem. Many traditional methods are no longer applicable. Moreover, the choice of the parameter q is a challenging problem. Recently many practical methods have been developed to deal with the minimization problem (1.3) in [5,12,18,24,26]. Among all L_q regularizations with $q \in (0, 1)$, the $L_{1/2}$ regularization, as a representative, was showed by Z.B. Xu et al. [25]. The formulae for $L_q(q = \frac{1}{2}, q = \frac{2}{3})$ regularization problems were deduced by Z.B. Xu et al. in [24] and [5], respectively. Krishnan et al. [18] and Z.B. Xu et al. [5] also demonstrated the advantages of $L_{1/2}$ and $L_{2/3}$ regularization in image restoration and compressed sensing, respectively.

In this paper, we propose a new robust and efficient adaptive algorithm for the minimization problem (1.3) with the operator $F \equiv \nabla$ and $L_q\left(q = \frac{1}{2}, 1, 2\right)$ regularization, by means of the operator splitting and our multigrid method [7,8,10]. The motivation of our paper is to use L_q norm of ∇u in TV-based regularization term and the adaptive combination of the three solutions for the $L_q(q = 2, 1, \frac{1}{2})$ norms in our method.

This paper is organized as follows. In Section 2, we briefly describe basic algorithms for the TV-based model problem (1.3) by using three norms $L_q\left(q = \frac{1}{2}, 1, 2\right)$, respectively. In Section 3, our adaptive algorithm of the three norms is presented. Section 4 is devoted to numerical experiments. We end this paper with a brief summary in Section 5.

2. Basic algorithms

In this section, we generalize the split Bregman method to the cases of three norms.

The following anisotropic problem is considered:

$$\arg\min_u ||u_x||_q^q + ||u_y||_q^q + \frac{\mu}{2}\|Ku - z\|_2^2. \tag{2.1}$$

First, the derivatives u_x and u_y are replaced by auxiliary variables d_x and d_y, respectively.

The formula of the problem (2.1) then becomes

$$(u^{k+1}, d^{k+1}) = \arg\min_{u,d_x,d_y} ||d_x||_q^q + ||d_y||_q^q + \frac{\mu}{2}\|Ku - z\|_2^2 + \frac{\lambda}{2}\|d_x - u_x\|_2^2 + \frac{\lambda}{2}\|d_y - u_y\|_2^2, \tag{2.2}$$

where the last two items are added such that the auxiliary variables d_x and d_y can be close to the derivatives u_x and u_y, respectively.

Using the idea of the split Bregman method, the minimization problem (2.2) is performed efficiently by iteratively minimizing with respect to u and d, separately. This yields the following three subproblems:

$$u^{k+1} = \arg\min_u \frac{\mu}{2}\left\|Ku - z + c^k\right\|_2^2 + \frac{\lambda}{2}\left\|d_x - u_x - b_x^k\right\|_2^2 + \frac{\lambda}{2}\left\|d_y - u_y - b_y^k\right\|_2^2, \tag{2.3}$$

$$d_x^{k+1} = \arg\min_{d_x} \|d_x\|_q^q + \frac{\lambda}{2}\left\|d_x - u_x^{k+1} - b_x^k\right\|_2^2, \tag{2.4}$$

$$d_y^{k+1} = \arg\min_{d_y} \|d_y\|_q^q + \frac{\lambda}{2}\left\|d_y - u_y^{k+1} - b_y^k\right\|_2^2, \tag{2.5}$$