Applied Mathematics and Computation 329 (2018) 315–324

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Faedo–Galerkin approximation of second order nonlinear differential equation with deviated argument

M. Muslim

School of Basic Sciences, Indian Institute of Technology Mandi, Kamand, H.P. 175 005, India

ARTICLE INFO

MSC:
34K35
34K30
93B05
93C25

Keywords:
Second order differential equation with deviated argument
Cosine family of linear operators
Faedo–Galerkin approximation

ABSTRACT

In this manuscript, we consider a second order nonlinear differential equation with deviated argument in a separable Hilbert space X. We used the strongly continuous cosine family of linear operators and fixed point method to study the existence of an approximate solution of the second order differential equation. We define the fractional power of the closed linear operator and used it to prove the convergence of the approximate solution. Also, we prove the existence and convergence of the Faedo–Galerkin approximate solution. Finally, we give an example to illustrate the application of these abstract results.

1. Introduction

Many partial differential equations that arise in several problems connected with the transverse motion of an extensible beam, the vibration of hinged bars, and many other physical phenomena can be modeled as a second order abstract differential equation. So, it is quite significant to study the existence and approximation of solutions for such systems in some suitable abstract spaces.

We consider a second order nonlinear differential equation with deviated argument in a separable Hilbert Space X

$$\begin{aligned} &x''(t) + Ax(t) = f(t, x(t), x[h(x(t), t)]), \quad t \in (0, T], \\ &x(0) = x_0, \quad x'(0) = y_0, \end{aligned} \tag{1.1}$$

where $x : J(= [0, T]) \to X$ is the state function, A is a closed, positive definite, self-adjoint linear operator with domain $D(A)$ dense in X and has the pure point spectrum. Also, $-A$ is the infinitesimal generator of a strongly continuous cosine family of linear operators $(C(t))_{t\in R}$ on X. $f: J\times X\times X \to X$ and $h : X \times \mathbb{R}^+ \to \mathbb{R}^+$ are suitable functions to be specified later.

Problems of existence of solution of second-order nonlinear systems represented by differential and integro-differential equations with classical initial conditions in abstract spaces have been investigated extensively by many authors. For the details on second order differential equations, we refer to [1–11] and references cited in these papers. A useful tool for the study of abstract second order differential equations is the theory of strongly continuous cosine families of operators. We refer the reader to [4–6,8,9] for the necessary concepts about cosine functions.

Initial studies concerning existence, uniqueness and finite-time blow-up of solutions for the following equation

$$\begin{aligned} u'(t) + Au(t) &= g(u(t)), \quad t \ge 0, \\ u(0) &= \phi, \end{aligned}$$

E-mail address: muslim@iitmandi.ac.in

https://doi.org/10.1016/j.amc.2018.01.060
0096-3003/

have been considered by Segal [12], Murakami [13] and Heinz and Von Wahl [14]. Bazley [10,11] has considered the following semilinear wave equation

$$u''(t) + Au(t) = g(u(t)), \quad t \geq 0, \qquad (1.2)$$
$$u(0) = \phi, \quad u'(0) = \psi,$$

and has established the uniform convergence of approximations of solutions to (1.2) using the results of Heinz and von Wahl [14]. Goethel [15] has proved the convergence of approximations of solutions to (1.2) but assumed g to be defined on the whole of H.

In my knowledge, Gal [16] was the first person who has considered the nonlinear abstract differential equations of order one with deviated arguments and study the existence and uniqueness of solutions by using the semigroup of linear operators. After the Gal [16] paper some authors [17–20] have worked on different types of abstract differential equations with deviated arguments. Several authors [18–24] studies the existence and convergence of approximate solutions of abstract differential equations of order one by using the analytic semigroup of linear operators in a separable Hilbert space.

There are only few papers [10,11,15] discussing the approximation of second order differential equations in infinite dimensional spaces. In order to fill this gap, we consider a nonlinear system described by a second order differential equation with deviated arguments in a separable Hilbert space and used the cosine family of linear operators to study the existence and convergence of the approximate solutions.

The work of this manuscript is motivated by Bazley [10]. We use the ideas of Bazley [10], Miletta [21] and Muslim [22] to establish the existence and convergence of finite dimensional approximate solution of Eq. (1.1).

The plan of the paper is as follows. In the first and second section, we give the introduction and provide some of the notations and results required for later sections. In the third section, we study the existence of approximate solution and Section 4 deal with the convergence of the approximate solutions obtained in Section 3. In Section 5, we study the existence and convergence of Faedo–Galerkin approximate solutions and in the last section, we have given an example to show the application of these results.

2. Preliminaries and assumptions

Throughout the paper, we assume that $-A$ be the infinitesimal generator of a strongly continuous cosine family, $(C(t))_{t\in R}$ of bounded linear operators defined on a separable Hilbert space X endowed with a norm $||.||$.

Definition 2.1. A one parameter family $(C(t))_{t\in\mathbb{R}}$ of bounded linear operators mapping the Banach space X into itself is called a strongly continuous cosine family if and only if

(i) $C(s+t)+C(s-t) = 2C(s)C(t)$ for all $s, t \in \mathbb{R}$.
(ii) $C(0) = I$.
(iii) $C(t)x$ is continuous in t on $\mathbb{R}$ for each fixed point $x\in X$.

$(S(t))_{t\in R}$ be the sine function associated to the strongly continuous cosine family, $(C(t))_{t\in R}$ which is defined by

$$S(t)x = \int_0^t C(s)x\,ds, \quad x \in X, \quad t \in R.$$

Let M and $\tilde{M}$ are positive constants such that $||C(t)|| \leq M$ and $||S(t)|| \leq \tilde{M}$ for every $t\in J$. $D(A)$ be the domain of the operator A which is defined by

$$D(A) = \{x \in X : C(t)x \text{ is twice continuously differentiable in t}\}.$$

$D(A)$ is the Banach space endowed with the graph norm $||x||_A = ||x|| + ||Ax||$ for all $x \in D(A)$. We define a set

$$E = \{x \in X : C(t)x \text{ is once continuously differentiable in t}\}$$

which is a Banach space endowed with norm $||x||_E = ||x|| + \sup_{0\leq t\leq 1} ||AS(t)x||$ for all $x\in E$.

With the help of $C(t)$ and $S(t)$, we define an operator valued function

$$\bar{h}(t) = \begin{bmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{bmatrix}.$$

$\bar{h}(t)$ is a strongly continuous group of bounded linear operators on the space $E\times X$ generated by the operator

$$\bar{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$$

defined on $D(A)\times E$. $AS(t)\colon E\to X$ is a bounded linear operator and that $AS(t)x \to 0$ as $t \to 0$, for each $x\in E$. If $x\colon [0, \infty)\to X$ is locally integrable function then

$$y(t) = \int_0^t S(t-s)x(s)ds$$