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A reduced-order extrapolated finite difference iterative scheme based on POD method for 2D Sobolev equation[☆]

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ABSTRACT

In this study, we devote ourselves to the reduced-order extrapolated finite difference iterative (ROEFDI) modeling and analysis for the two-dimensional (2D) Sobolev equation. To this end, we first establish the reduced-order extrapolated finite difference iterative (ROEFDI) scheme holding sufficiently high accuracy but containing very few degrees of freedom for the 2D Sobolev equation via the proper orthogonal decomposition (POD) technique. And then, we analyze the stability and convergence of the ROEFDI solutions. Finally, we use the numerical experiments to verify the feasibility and effectiveness of the ROEFDI scheme.

1. Introduction

As a matter of convenience and without loss of universality, we consider the following two-dimensional (2D) Sobolev equation.

$$\begin{cases} \dfrac{\partial u}{\partial t} - \varepsilon \dfrac{\partial \Delta u}{\partial t} - \gamma \Delta u = f(x, y, t), & (x, y, t) \in \Omega \times (0, T), \\ u(x, y, t) = Q(x, y, t), & (x, y, t) \in \partial\Omega \times (0, T], \\ u(x, y, 0) = G(x, y), & (x, y) \in \Omega, \end{cases} \tag{1}$$

where $\Omega = (a, b) \times (c, d) \subset \mathbb{R}^2$ is the bounded open set with the boundary $\partial\Omega$, $u(x, y, t)$ is the unknown function, ε and γ are two known positive parameters, and $f(x, y, t)$ and $Q(x, y, t)$ as well as $G(x, y)$ are three given functions.

The existence and uniqueness of the analytic solution for the Sobolev equation (1) had been proved in [6,9].

The Sobolev equation plays an extremely important role in many numerical simulations of mathematical physics problems such as the fluid seepage through fractured rock or soil [2], the heat exchange in different media [32], and the moisture migration in soil [27]. However, because the Sobolev equation usually has complex known data or computational domains in the actual engineering applications, even if theoretically there exists the analytical solution, it can not be generally sought out so that one has to rely on the numerical methods. In nearly 40 years, the Sobolev equation has been closely watched,

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there have been many numerical research reports (see, e.g., [3,8,10,15,16,18,31,34]). Among all numerical methods, the finite difference (FD) scheme is considered to be one of the simplest and most convenient as well as the most easily programming calculating numerical methods for solving the 2D Sobolev equation [31]. However, the classical FD scheme for the 2D Sobolev equation is a macroscale system of equations including lots of unknowns, i.e., degrees of freedom so as to undertake very large computational load in the real-life engineering applications. Therefore, a key issue is how to lessen the unknowns of the classical FD scheme so as to alleviate the truncated error accumulation in the numerical calculating procedure and save the computing time but keeping sufficiently high accuracy numerical solutions.

A large number of numerical examples (see, e.g., [4,7,11,13,14,19,22–26,28,29,30]) have proved that the proper orthogonal decomposition (POD) technique is a very resultful approach to lessen the degrees of freedom for numerical models and alleviate the truncated error accumulation in the numerical calculating procedure. The POD method has also played an important role in the reduced-basis of parametric models for the optimal control problems of nonlinear PDEs [12]. But the most existing reduced-order models as mentioned above were established via the POD basis of the classical numerical solutions at all time nodes, before calculating the reduced-order numerical solutions at the same time nodes, which were some worthless repeated calculations. Since 2014, some reduced-order extrapolating FD schemes based on the POD technique for PDEs have been established successively by Luo's team (see, e.g., [1,17,20,21]) in order to avoid the worthless repeated calculations.

However, as far as we know, there has been not any study that the POD technique is used to reduce the degrees of freedom in the classical FD scheme for the 2D Sobolev equation. Therefore, in this article, we extend the approaches in [1,17,20,21] to the 2D Sobolev equation, establishing a reduced-order extrapolated finite difference iterative (ROEFDI) scheme containing very few unknowns but holding sufficiently high accuracy by the POD technique, analyzing the stability and convergence of the ROEFDI solutions, and verifying the feasibility and effectiveness of the ROEFDI scheme via numerical experiments.

The major difference between the ROEFDI scheme and the existing POD-based reduced-order extrapolating FD schemes (see, e.g., [1,17,20,21]) consists in that the Sobolev equation not only includes the time first order derivative term and the spacial variables 2nd order derivative terms but also contains a mixed derivative term about time first order and spacial variables 2nd order so that either the establishment of the ROEFDI scheme or the analysis of the stability and convergence of the ROEFDI solutions faces more difficulties and needs more skills than the existing reduced-order extrapolating FD schemes as mentioned, but the Sobolev equation has some specific applications. Fortunately, we adopt the analysis technique of the vector and matric for the stability and convergence of the classical FD and ROEFDI solutions such that the theoretical analysis not only becomes very simple and convenient but the numerical simulations in computer can also easily implement. Especially, the ROEFDI scheme only uses the few classical FD solutions on the initial very short time span to constitute the POD basis and establish the ROEFDI scheme. Therefore, it also has not repeated calculation like References [1,17,20,21]. Hence, it is development and improvement over the existing those as mentioned above.

The rest part of the article is arranged as follows. The classical FD scheme for the 2D Sobolev equation is posed in Section 2. The ROEFDI scheme based on the POD technique for the 2D Sobolev equation is established in Section 3. The stability and convergence of the ROEFDI solutions are analyzed in Section 4. In Section 5, some numerical experiments are presented to verify the feasibility and effectiveness of the ROEFDI scheme. Finally, some main conclusions are summarized in Section 6.

2. The classical FD scheme for the 2D Sobolev equation

Let Δt be the time step and Δx and Δy be, separately, the spacial steps in x and y directions, $u_{i,j}^n$ denote the classical FD approximations of u at points (x_i, y_j, t_n) $(x_i = a + i\Delta x,\ y_j = c + j\Delta y,\ t_n = n\Delta t,\ 0 \leqslant i \leqslant I \equiv [(b-a)/\Delta x],\ 0 \leqslant j \leqslant J \equiv [(d-c)/\Delta y]$, and $0 \leq n \leq N \equiv [T/\Delta t]$, where $[e]$ represents the integer part of the real number e).

By approximating to the derivatives of (1) by means of the following difference quotient:

$$\frac{\partial u}{\partial t} = \frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} + O(\Delta t) \approx \frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t},$$
$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{j+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{\Delta x^2} + O(\Delta x^2) \approx \frac{u_{j+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{\Delta x^2},$$
$$\frac{\partial^2 u}{\partial y^2} = \frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{\Delta y^2} + O(\Delta y^2) \approx \frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{\Delta y^2},$$

we obtain the following classical FD scheme:

$$\begin{aligned} & u_{i,j}^{n+1} - \frac{\varepsilon}{\Delta x^2}(u_{i+1,j}^{n+1} - 2u_{i,j}^{n+1} + u_{i-1,j}^{n+1}) - \frac{\varepsilon}{\Delta y^2}(u_{i,j+1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j-1}^{n+1}) \\ & \quad = (\gamma\Delta t - \varepsilon)\left[\frac{1}{\Delta x^2}(u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n) + \frac{1}{\Delta y^2}(u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n)\right] + u_{i,j}^n + \Delta t f_{i,j}^n, \\ & \quad i = 1, 2, \dots, I-1,\ \ j = 1, 2, \dots, J-1,\ \ n = 0, 1, 2, \dots, N-1 \end{aligned} \tag{2}$$