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On acyclically 4-colorable maximal planar graphs

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ABSTRACT

An acyclic coloring of a graph is a proper coloring of the graph, for which every cycle uses at least three colors. Let $\mathcal{G}^4$ be the set of maximal planar graphs of minimum degree 4, such that each graph in $\mathcal{G}^4$ contains exactly four odd-vertices and the subgraph induced by the four odd-vertices contains a quadrilateral. In this article, we show that every acyclic 4-coloring of a maximal planar graph with exact four odd-vertices is locally equitable with regard to its four odd-vertices. Moreover, we obtain a necessary and sufficient condition for a graph in $\mathcal{G}^4$ to be acyclically 4-colorable, and give an enumeration of the acyclically 4-colorable graphs in $\mathcal{G}^4$.

1. Introduction

All graphs considered in this paper are simple and finite. For a graph G, let $V(G)$ and $E(G)$ be the set of*vertices* and *edges* of G respectively. A *neighbor* of a vertex v in G is a vertex that is connected to v by an edge. We denote by $N_G(u)$ the set of neighbors in G of u, by $d_G(u) = |N_G(u)|$ the *degree* in G of u, and by $\delta(G)$ and $\Delta(G)$ the minimum and maximum degree of G, respectively. A vertex u with $d_G(u) = k$ is called a *k-vertex* of G, and an *odd-vertex* when k is odd and an *even-vertex* when k is even. Define $N_G[u] = N_G(u) \cup \{u\}$. For $V' \subseteq V(G)$, $G[V']$ is the subgraph of G induced by V', and $G - V'$ is the graph obtained from G by deleting vertices in V' and the edges incident with them. A k-cycle C of a connected graph G is called a *separating k-cycle* if $G - V(C)$ results in a disconnected graph, where a *k-cycle* is a cycle of length k. For more notations and terminologies, we refer the reader to the book [2].

A proper *k-coloring* of a graph G is a partition $\{V_1, V_2, \ldots, V_k\}$ of $V(G)$, where V_i is an independent set for $i = 1, 2, \ldots, k$ and denotes the (possibly empty) set of vertices assigned color i. The sets V_i are called the *color classes* of the coloring. An *acyclic k-coloring* of a graph G is a proper k-coloring of G such that every cycle uses at least three colors. G is called *acyclically k-colorable* if it admits an acyclic k-coloring.

There are a large number of applications of acyclic colorings. For example, acyclic colorings of graphs can be applied to estimate large and sparse symmetric matrices [9,12], and to compute upper bounds on the volume of 3-dimensional straight-line grid drawings of planar graphs [10]. The acyclic chromatic number of a graph can be used to obtain an upper bound on the size of a "feedback vertex set" of a graph, which has wide applications in operation system, database system, genome assembly, and VLSI chip design [11]. Additionally, acyclic coloring has also found applications to some other plane graph coloring and partitioning problems. In particular, acyclic 5-colorability implies, by means of short nice arguments,

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the following best-known upper bounds for coloring parameters on planar graphs: 80 for the oriented chromatic number [16] and, combined with the Four Color Theorem, 20 for the star chromatic number [1].

Acyclic coloring was introduced by Grünbaum et al. [13], who proved that every planar graph is acyclically 9-colorable, and conjectured that five colors are sufficient. Borodin [3] (also see [4]) established the validity of this conjecture by showing that every planar graph is acyclically 5-colorable. This bound is the best because there exist planar graphs with no acyclic 4-colorings [13,17]. In 1976, Kostochka and Mel'nikov [14] proved that graphs with no acyclic 4-coloring can be found among 3-degenerated bipartite planar graphs. Additionally, with regard to the acyclically 4-colorable planar graphs, many sufficient conditions have been obtained [5–8,15], in which the best result is given by Borodin who showed that each planar graph without 4- and 5-cycles is acyclically 4-choosable [7].

A planar graph G is called a *maximal planar graph* (or *plane triangulation*) if the addition of any edge to G results in a nonplanar graph. In what follows, we denote by $\mathcal{M}^4$ the set of maximal planar graphs with exactly four odd-vertices, and by $\mathcal{G}^4$ a subclass of $\mathcal{M}^4$ such that each $G \in \mathcal{G}^4$ has minimum degree 4 and the subgraph of G induced by its four odd-vertices contains a quadrilateral. Further, we use $\mathcal{G}^4_n$ to denote the set of graphs on n vertices in $\mathcal{G}^4$.

In [18], the authors proved that any acyclically 4-colorable maximal planar graph of minimum degree 4 contains at least four odd-vertices, and gave some necessary conditions for a 4-connected maximal planar graph with exactly four odd-vertices to be acyclically 4-colorable. It seems difficult to find the sufficient conditions for a 4-connected graph in $\mathcal{M}^4$ to be acyclically 4-colorable. In this paper, we show that every acyclic 4-coloring of an acyclically 4-colorable maximal planar graph with exactly four odd-vertices is locally equitable with regard to its four odd-vertices. Moreover, we obtain a necessary and sufficient condition for a graph $G \in \mathcal{G}^4$ to be 4-colorable and give an enumeration formula to compute the number of 4-colorable maximal planar graphs in $\mathcal{G}^4_n$.

2. Acyclic 4-colorings of maximal planar graphs with exactly four odd-vertices

This section is devoted to the structure of acyclic 4-colorings of maximal planar graphs with exactly four odd-vertices.

For a k-coloring f of a graph G and a vertex set $V' \subseteq V(G)$, we refer to f as a *locally equitable coloring with regard to* V' if $|V_i \cap V'| = |V_j \cap V'|$ for any two distinct color classes V_i and V_j of f with $V_i \cap V' \neq \emptyset$ and $V_j \cap V' \neq \emptyset$. We also say V' to be *colored* (*locally*)*equitably* under f.

The dual graph G^* of a plane graph G is a graph that has a vertex corresponding to each face of G, and an edge joining two neighboring faces for each edge in G. If a graph $G \in \mathcal{M}^4$, we can easily see that the dual graph G^* of G is a planar cubic 3-connected graph that contains exactly four odd-faces, where an *odd-face* of a planar graph is a face that the number of edges in its boundary is odd number.

Let X and Y be the sets of vertices of a planar graph $G = (V, E)$ such that $X = V \setminus Y$. We refer to the set of edges of G with one end in X and the other end in Y, denote by $E[X, Y]$, as an *edge cut* of G. A natural conclusion follows that $G^*[E^*[X, Y]]$ is a cycle for any edge cut $E[X, Y]$ ($|E[X, Y]| \geq 3$) of G, where G^* is the dual of G and $E^*[X, Y]$ is the set of edges corresponding to $E[X, Y]$ in G^*.

For a maximal planar graph $G \in \mathcal{M}^4$, if $V(G)$ has a partition $\{V_1, V_2\}$ such that $G[V_i]$ is a tree for $i = 1, 2$, then we desire to know how many odd-vertices are contained in V_1 and V_2 respectively. We begin with a general observation as follows.

Recall that a cycle containing all vertices of a graph is called a *Hamilton cycle* of the graph.

Lemma 2.1. *Let G be a maximal planar graph. If there is a partition $\{V_1, V_2\}$ of $V(G)$ such that $G[V_1]$ and $G[V_2]$ are trees, then V_i, $i = 1, 2$, contains an even number of odd-vertices.*

Proof. Let $E[V_1, V_2]$ be an edge cut of G, and $E^*[V_1, V_2]$ be the corresponding edge set of $E[V_1, V_2]$ in the dual G^* of G. Then $G^*[E^*[X, Y]]$ is a cycle, say C. Since both $G[V_1]$ and $G[V_2]$ are trees and each face of G is a triangle, it follows that each face of G contains exactly two edges of $E[V_1, V_2]$. Therefore, C is a Hamilton cycle of G^*. Let F_i be the set of faces in G^* corresponding to V_i for $i = 1, 2$. Then, F_1 and F_2 are in the interior and exterior of C, respectively. Because the number of odd-faces in a planar graph is even and C contains even number vertices (since G^* is 3-regular), it has that both F_1 and F_2 contain an even number of odd-faces. Correspondingly, V_1 and V_2 contains an even number of odd-vertices. □

According to Lemma 2.1, we have the following result.

Corollary 2.2. *For $G \in \mathcal{M}^4$, if there is a partition $\{V_1, V_2\}$ of $V(G)$ such that $G[V_i]$ is a tree for $i = 1, 2$, then either each of V_1 and V_2 contains exactly two odd-vertices, or one of V_1, V_2 contains four odd-vertices.*

For a $k(\geq 2)$-coloring f of a graph G, we use $G[i, j]$ ($i \neq j$) to denote the subgraph of G, induced by the vertices colored by i and j under f.

Theorem 2.3. *Let $G \in \mathcal{M}^4$ be an acyclically 4-colorable maximal planar graph, and v_1, v_2, v_3, v_4 be its four odd-vertices. Then for each acyclic 4-coloring f of G, $\{v_1, v_2, v_3, v_4\}$ are colored equitably under f.*

Proof. Let $C = \{1, 2, 3, 4\}$ be the color set. Since f is an acyclic 4-coloring of G, it follows that $G[i, j]$ does not contain any cycle for $i, j \in C$, $i \neq j$. So, $|V(G[i, j])| - |E(G[i, j])| \geq 1$. Because $3|V(G)| - |E(G)| = 6$ by the Euler formula, we can easily deduce that $|V(G[i, j])| - |E(G[i, j])| = 1$. Hence $G[i, j]$ is a tree. Consider $G[1, 2]$ and $G[3, 4]$; by Corollary 2.2, either $G[1, 2]$ contains two odd-vertices and $G[3, 4]$ contains two odd-vertices, or one of $G[1, 2]$ and $G[3, 4]$ contains four odd-vertices