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On graphs whose Wiener complexity equals their order and on Wiener index of asymmetric graphs

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ABSTRACT

If u is a vertex of a graph G, then the transmission of u is the sum of distances from u to all the other vertices of G. The Wiener complexity $C_W(G)$ of G is the number of different complexities of its vertices. G is transmission irregular if $C_W(G) = n(G)$. It is proved that almost no graphs are transmission irregular. Let T_{n_1,n_2,n_3} be the tree obtained from paths of respective lengths n_1, n_2, and n_3, by identifying an end-vertex of each of them. It is proved that T_{1,n_2,n_3} is transmission irregular if and only if $n_3 = n_2 + 1$ and $n_2 \notin \left\{(k^2-1)/2, (k^2-2)/2\right\}$ for some $k \geq 3$. It is also proved that if T is an asymmetric tree of order n, then the Wiener index of T is bounded by $(n^3 - 13n + 48)/6$ with equality if and only if $T = T_{1,2,n-4}$. A parallel result is deduced for asymmetric uni-cyclic graphs.

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1. Introduction

If $G = (V, E)$ is a graph and $u \in V$, then the *transmission* of u (also known as the *distance* of u or the *remoteness* of u) is defined as the sum of distances from u to all the other vertices of G, that is, $\mathrm{Tr}_G(u) = \sum_{x \in V} d_G(u, x)$, see [1,26,28]. Here $d_G(u, x)$ is the standard shortest path distance between u and x in G. We will shortly write $\mathrm{Tr}(u)$ and $d(u, x)$, whenever G will be clear from the context. The *Wiener index* $W(G)$ of G is the sum of the distances between all pairs of vertices of G, in other words, $W(G) = \frac{1}{2}\sum_{u \in V(G)} \mathrm{Tr}(u)$. The papers [11,16,25,30] are some classical references on the Wiener index; to see the extent to which it has already been investigated we refer to surveys [9,10,20] and selected recent papers [4,7,8,13,15,17,18,22,24,29]. Transmission of a vertex is also important elsewhere, notably in location theory where vertices with extremal transmission are of interest because they are target locations for facilities, cf. [5,21,27].

The fact that the sum of the transmissions of all the vertices of G is twice the Wiener index of G, led in [2] to the introduction of the *Wiener complexity* $C_W(G)$ as the number of different transmissions in G. Actually, C_W was named *Wiener dimension* and denoted $\dim_W$ in [2], but here we rather follow the notation and terminology of a general approach from [3]. In this approach, the *I-complexity* C_I of an arbitrary summation-type topological index I is defined as the number of different contributions to I in its summation formula.

Call a graph G to be *transmission regular* [23] if all its vertices have the same transmission and *transmission irregular* if its vertices have pairwise different transmissions. In other words, transmission irregular graph are the graphs that have

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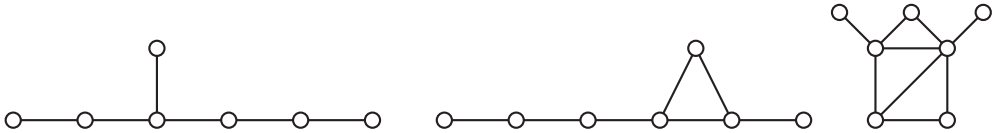

Fig. 1. Sporadic examples of transmission irregular graphs.

the largest possible Wiener complexity over all graphs of a given order. The reader can easily verify that each of the three graphs drawn in Fig. 1 is a transmission irregular graph.

In this paper we are interested in transmission irregular graphs and proceed as follows. In the rest of the section terminology and notation needed are stated. In the next section we first show that almost all graphs are not transmission irregular. Then we introduce the trees T_{n_1,n_2,n_3} and characterize transmission irregular trees among the trees T_{1,n_2,n_3}. We also prove that if a graph G has at least three vertices of the same degree, then G and its complement cannot be both transmission irregular. In Section 3 we prove two extremal results on the Wiener index of asymmetric trees and of asymmetric uni-cyclic graphs and deduce corresponding consequences for transmission irregular graphs. In particular, if T is a transmission irregular tree of order n and $W(T) = \left(n^3 - 13n + 48\right)/6$, then $T = T_{1,2,3}$.

All graphs considered in this paper are connected. The order of a graph G is denoted with $n(G)$ and the degree of $u \in V(G)$ with $\deg(u)$. The *distance* $d_G(u,v)$ ($d(u,v)$ for short) is the number of edges on a shortest u,v-path. The *eccentricity* $\mathrm{ecc}_G(u)$ ($\mathrm{ecc}(u)$ for short) of a vertex u is the maximum distance between u and the other vertices. The *diameter* $\mathrm{diam}(G)$ of G is the maximum eccentricity of its vertices. The automorphism group of G is denoted with $\mathrm{Aut}(G)$. A graph is *asymmetric* if $|\mathrm{Aut}(G)| = 1$. Finally, for a positive integer n we use the notation $[n] = \{1, \ldots, n\}$.

2. Transmission irregular graphs are rare

An automorphism of a graph preserves the distance function. Hence, if u and v are vertices of a graph G such that $\alpha(u) = v$ holds for some $\alpha \in \mathrm{Aut}(G)$, then $\mathrm{Tr}(u) = \mathrm{Tr}(v)$. It follows that a transmission irregular graph is asymmetric and, as it well known, almost all graphs are asymmetric [12]. On the other hand, the fraction of transmission irregular graphs among asymmetric graphs is small as the next result asserts.

Theorem 2.1. *Almost all graphs are not transmission irregular.*

Proof. It is well known that almost every graph has diameter 2. Hence the conclusion of the theorem will follow after proving that a transmission irregular graph $G = (V, E)$ has $\mathrm{diam}(G) \geq 3$. Suppose then that $\mathrm{diam}(G) = k$. Clearly, $k \neq 1$ as complete graphs are not transmission irregular. Suppose that $k = 2$. Then for any vertex $u \in V$ we have $\mathrm{Tr}(u) = \deg(u) + 2(n(G) - 1 - \deg(u)) = 2n(G) - 2 - \deg(u)$. Since at least two vertices are of the same degree, they have the same transmission. We conclude that $k \geq 3$. □

Transmission irregular graphs are thus rare. To increase the family of known such graphs, motivated by the leftmost transmission irregular graph from Fig. 1, we introduce the following family of trees. Let n_1, n_2, n_3 be integers for which $1 \leq n_1 < n_2 < n_3$ holds. Then the tree T_{n_1,n_2,n_3} has the vertex set

$$\{u\} \cup \{x_1, \ldots, x_{n_1}\} \cup \{y_1, \ldots, y_{n_2}\} \cup \{z_1, \ldots, z_{n_3}\},$$

and the edge set

$$\{ux_1, x_1x_2, \ldots, x_{n_1-1}x_{n_1}\} \cup \{uy_1, y_1y_2, \ldots, y_{n_2-1}y_{n_2}\} \cup \{uz_1, z_1z_2, \ldots, z_{n_3-1}z_{n_3}\}.$$

The tree T_{n_1,n_2,n_3} can be described as the graph obtained from paths of respective lengths n_1, n_2, and n_3, by identifying an end-vertex of each of them. Note that $T_{1,\,2,\,3}$ is the leftmost transmission irregular graph from Fig. 1. Ideally, we would like to characterize transmission irregular trees among the trees T_{n_1,n_2,n_3}. For this sake it is not difficult to deduce the following equations:

$$\mathrm{Tr}(x_i) = \binom{n_1 - i + 1}{2} + \binom{n_2 + i + 1}{2} + \binom{n_3 + i + 1}{2} - \binom{i+1}{2}, \tag{1}$$

$$\mathrm{Tr}(y_j) = \binom{n_2 - j + 1}{2} + \binom{n_1 + j + 1}{2} + \binom{n_3 + j + 1}{2} - \binom{j+1}{2}, \tag{2}$$

$$\mathrm{Tr}(z_k) = \binom{n_3 - k + 1}{2} + \binom{n_1 + k + 1}{2} + \binom{n_2 + k + 1}{2} - \binom{k+1}{2}, \tag{3}$$

$$\mathrm{Tr}(u) = \binom{n_1 + 1}{2} + \binom{n_2 + 1}{2} + \binom{n_3 + 1}{2}, \tag{4}$$

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