Applied Mathematics and Computation 329 (2018) 420–431

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Predetermining the number of periodic steps in multi-step Newton-like methods for solving equations and systems of equations

I.K. Argyros[a,*], St. Măruşter[b]

[a] Department of Mathematical Science, Cameron University, Lawton, OK 73505, USA
[b] West University of Timişoara, Department of Mathematics and Computer Science, Timişoara 30174, Romania

ARTICLE INFO

MSC:
45G10
47H17
65J15
65G99

Keywords:
Multi-step Newton-type method
Banach space
Euclidean space
Semi-local convergence
Center-Lipschitz and Lipschitz conditions
Systems of differential equations

ABSTRACT

The goal of this paper is to predetermine the steps m after which the first derivative is re–evaluated for multi-step Newton-type method used to approximate solutions of equations.

1. Introduction

Numerous problems in computational sciences and also in engineering such as, problems from optimization; economic equilibrium theory; astrophysics; elasticity; neutron transport; dynamical systems to mention a few, can be brought in a form like

$$F(x) = 0 \tag{1.1}$$

using mathematical modeling [1,4–6,11–13]. Here, $F\colon D\subseteq B_1 \to B_2$ is a Fréchet differentiable operator, B_1, B_2 are Banach spaces and D is a convex subset of B_1. Finding a solution p of Eq. (1.1) is a great challenge in general. One wishes to obtain a solution p in closed form but this is possible only in spacial cases. That explains why most solution methods for Eq. (1.1) are usually iterative.

There is a plethora of high convergence order iterative methods for solving Eq. (1.1) involving high order derivatives [1–15]. However, these methods are unattractive, since they are very expensive. Currently, there is an increasing interest in developing iterative methods of high convergence order using the first Fréchet derivative and and its inverse. In particular, we consider the k-step Newton-type method defined for each $n = 0, 1, 2, \ldots, k$ where k is a natural number, and some starting point $x_0 = x_0^0 \in D$ by

* Corresponding author.
E-mail addresses: iargyros@cameron.edu, ioannisa@cameron.edu (I.K. Argyros), maruster@info.uvt.ro (St. Măruşter).

https://doi.org/10.1016/j.amc.2018.02.024
0096-3003/

$$
\begin{aligned}
x_n^1 &= x_n^0 - F'(x_n^0)^{-1}F(x_n^0)\\
x_n^2 &= x_n^1 - F'(x_n^0)^{-1}F(x_n^1)\\
&\vdots\\
x_n^k &= x_n^{k-1} - F'(x_n^0)^{-1}F(x_n^{k-1})\\
x_{n+1}^0 &= x_n^k
\end{aligned}
\tag{1.2}
$$

The convergence of such methods has been studied under Lipschitz-conditions and w-type conditions by several authors [4–10]. The convergence order is $k+1$. The convergence domain of method (1.2) is small in general under the mentioned conditions limiting the applicability of method (1.2). In the present study, we use center-Lipschitz conditions for the computation of the upper bound on the inverses $\|F'(x_n^0)^{-1}F'(x_0^0)\|$ instead of the less accurate Lipschitz conditions. Moreover, we locate a more precise domain than before, where the iterates lie leading to smaller Lipschitz constants. We also use our method of recurrent functions that has been effective in the study of iterative methods [1–4]. This way, we obtain the following advantages (I):

(i_1) Larger convergence domain.
(i_2) Tighter error bounds on the distances $\|x_m^m - x_n^{m-1}\|$, $\|x_n^n - p\|$, $m = 2, 3, \ldots, k$.
(i_3) More precise information on the location of the solution p.

We also provide some results on how to predetermine the choice of k in the general case of a Banach space setting as well as in the special case when $B_1 = B_2 = \mathbb{R}^j$ (j a natural number).

The rest of the paper is structured as follows: The semi-local convergence analysis is presented in Section 2 and the applications in Section 3.

2. Semi-local convergence analysis

We need an auxiliary result on majorizing sequences for method (1.2).

Lemma 1. *Let L_0, L and η be positive numbers with $L_0 \le L$. Define the scalar sequence $\{t_{n,m}\}$ for each $n = 0, 1, 2, \ldots,$ $m = 1, 2, \ldots, k$ by*

$$
t_{0,0} = 0,\ \ t_{0,1} = \eta,
$$

$$
t_{n,2} = t_{n,1} + \frac{L}{2}\frac{(t_{n,1} - t_{n,0})^2}{1 - L_0 t_{n,0}} \quad \text{for} \quad m = 2, \tag{2.1}
$$

and for $m \neq 2$

$$
t_{n,m} = t_{n,m-1} + \frac{L}{2}\frac{[(t_{n,m-1} - t_{n,0}) + (t_{n,m-2} - t_{n,0})](t_{n,m-1} - t_{n,0})}{1 - L_0 t_{n,0}}
$$

$$
t_{n,k} = t_{n+1,0}
$$

Denote by $\xi = \xi(k) = \xi_k$ the smallest positive root of the function φ_k defined on the interval [0, 1] *by*

$$
\varphi_k(t) = L(t^k - 1)[(1 + t + \cdots + t^{k-1}) + (1 + t + \cdots + t^{k-2})] + 2L_0(1 + t + \cdots + t^{k-1})t^{k+1}. \tag{2.2}
$$

Suppose that

$$
0 \le \frac{L(t_{0,k} + t_{0,k-1})}{2(1 - L_0 t_{0,k})} \le \xi < 1 - L_0\eta. \tag{2.3}
$$

Then, scalar sequence $\{t_{n,m}\}$ is increasing, bounded from above by

$$
t^{**} = \frac{\eta}{1 - \xi} \tag{2.4}
$$

and converges to its unique least upper bound t^* satisfying

$$
\eta \le t^* \le t^{**}. \tag{2.5}
$$

Moreover, for each $n = 0, 1, 2, \ldots,$ $m = 1, 2, \ldots, k,$ the following items hold

$$
t_{n,m-1} \le t_{n,m} \tag{2.6}
$$

$$
t^* < \frac{1}{L_0} \tag{2.7}
$$

$$
0 \le t_{n,m} - t_{n,m-1} \le \xi(t_{n,m-1} - t_{n,m-2}) \tag{2.8}
$$

and