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High-order numerical approximation formulas for Riemann-Liouville (Riesz) tempered fractional derivatives: construction and application (I)☆

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A B S T R A C T

In this paper, we develop a new numerical algorithm for solving the Riesz tempered space fractional diffusion equation. The stability and convergence of the numerical scheme are discussed via the technique of matrix analysis. Finally, numerical experiments are performed to confirm the effectiveness of our numerical algorithm.

1. Introduction

In recent years, some theories and experiments show that a large number of non-classical phenomena that appeared in nature can be described in a concise and fruitful way by the fractional calculus [2,15,23,24,29,32,33,36,44]. Nowadays, fractional calculus and some mathematic models with fractional derivatives have become a powerful tool to model the particle transport in anomalous diffusion in various fields due to their good mathematical features. In practical applications, several types of the usual fractional derivatives, such as the Caputo derivative, Riemann–Liouville derivative, and Riesz fractional derivative are introduced [19,29]. For the tempered fractional calculus, which can be obtained by multiplying an exponential factor to the integrand function of the usual fractional calculus, and it is commonly used in truncated exponential power law description [25,34,41].

From the mathematical point of view, it is well known that the continuum time random walk is a very important and common mathematical model for particle kinetics, which incorporates waiting times and/or non-Gaussian jump distributions with divergent second moments to account for the anomalous jumps called Lévy flights [23,42]. It produces that the second moment diverges, that is, $\langle x^2(t)\rangle = \infty$. The usually time and/or space fractional diffusion equation can be obtained by taking the continuous limit of such models [15,23]. However, from the point of view of an experimental study, many physical processes usually take place in bounded domains in finite times and have finite moments. Therefore, the divergent second

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Table 1
The absolute errors and convergence orders of Example 1 by the numerical differential formula (9) with $n = 2, \lambda = 1/500$.

α	h	The absolute errors	The convergence orders
1.2	$\frac{1}{10}$	1.055602e-002	–
	$\frac{1}{12}$	7.182623e-003	2.1118
	$\frac{1}{14}$	5.046779e-003	2.2894
	$\frac{1}{16}$	3.583863e-003	2.5635
	$\frac{1}{18}$	2.529330e-003	2.9587
1.4	$\frac{1}{10}$	1.539565e-002	–
	$\frac{1}{12}$	1.082498e-002	1.9319
	$\frac{1}{14}$	7.996273e-003	1.9648
	$\frac{1}{16}$	6.123688e-003	1.9981
	$\frac{1}{18}$	4.818992e-003	2.0342
1.5	$\frac{1}{10}$	1.745507e-002	–
	$\frac{1}{12}$	1.226204e-002	1.9368
	$\frac{1}{14}$	9.068450e-003	1.9572
	$\frac{1}{16}$	6.966119e-003	1.9751
	$\frac{1}{18}$	5.508809e-003	1.9927
1.6	$\frac{1}{10}$	1.925169e-002	–
	$\frac{1}{12}$	1.349002e-002	1.9507
	$\frac{1}{14}$	9.966179e-003	1.9640
	$\frac{1}{16}$	7.656185e-003	1.9747
	$\frac{1}{18}$	6.060484e-003	1.9844
1.8	$\frac{1}{10}$	2.142076e-002	–
	$\frac{1}{12}$	1.492405e-002	1.9821
	$\frac{1}{14}$	1.098727e-002	1.9866
	$\frac{1}{16}$	8.423419e-003	1.9900
	$\frac{1}{18}$	6.661189e-003	1.9928

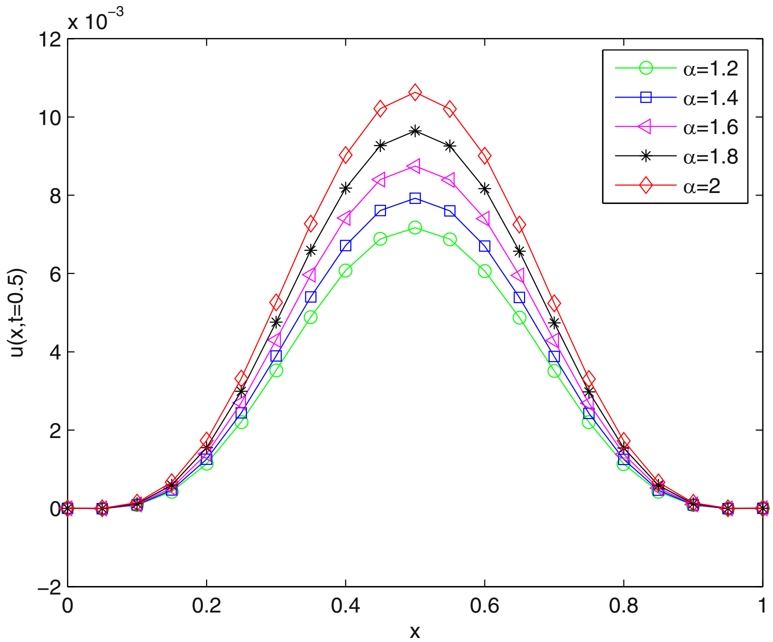

Fig. 1. The comparison of the numerical solution with the exact solution for different order α by the numerical algorithm (13) with $\lambda = 1/100$, $\tau = 1/100$, $h = 1/20$.

moments are not available for such processes. In order to overcome these barriers, there are many different techniques are employed, such as in [26], the truncated Lévy flights are developed by discarding the very large jumps. Two other ways are proposed by Sokolov et al. [31], who add a high-order power-law factor, and Chechkin et al. [7], who add a nonlinear friction term, which also leads to the finite second moments. At present, the most effective and popular method is exponentially tempering the probability of large jumps of Lévy flights, which results in tempered-stable Lévy processes with finite moments, and the corresponding tempered fractional differential equations are constructed [2,34]. It's worth