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On the [1,2]-domination number of generalized Petersen graphs

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ABSTRACT

A *dominating set* in a graph $G=(V,E)$ is a subset S of V such that $N[S]=V$, that is, each vertex of G either belongs to S or is adjacent to at least one vertex in S. The minimum cardinality of a dominating set in G is called the *domination number*, denoted by $\gamma(G)$. A subset S of V is a *[1,2]-set* if, for every vertex $v\in V\setminus S$, v is adjacent to at least one but no more than two vertices in S. The *[1,2]-domination number* of a graph G, denoted by $\gamma_{[1,2]}(G)$, is the minimum cardinality of a [1, 2]-set of Chellali et al. gave some bounds for $\gamma_{[1,2]}(G)$ and proposed the following problem: which graphs satisfy $\gamma(G)=\gamma_{[1,2]}(G)$. Ebrahimi et al. determined the exact value of the domination number for generalized Petersen graphs $P(n, k)$ when $k\in\{1,2,3\}$. In this paper, we determine the exact values of $\gamma_{[1,2]}(P(n,k))$ for $k\in\{1,2,3\}$. We also show that $\gamma_{[1,2]}(P(n,k))=\gamma(P(n,k))$ for $k=1$ and $k=3$, respectively, while for $k=2$, $\gamma_{[1,2]}(P(n,k))\neq\gamma(P(n,k))$ except for $n=6,7,9,12$.

1. Introduction

In this paper, all graphs we considered are simple, finite and undirected. We refer to [1] for the undefined notation and terminology.

Let $G=(V,E)$ be a graph. For $v\in V$, the *open neighborhood* of v, denoted by $N(v)$, is the set of all vertices adjacent to v. Denote by, $N[v]=\{v\}\cup N(v)$, the *closed neighborhood* of v. For any vertex set $S\subseteq V$, the *open neighborhood* of S is defined as $N(S)=\bigcup_{v\in S}N(v)$, while the *closed neighborhood* of S is the set $N[S]=\bigcup_{v\in S}N[v]$.

A *dominating set* in G is a subset S of V such that $N[S]=V$, that is, each vertex of G either belongs to S or is adjacent to at least one vertex in S. The minimum cardinality of a dominating set in G is called the *domination number* of G and denoted $\gamma(G)$. Dominating set problems have many applications in numerous facility location problems and social networks. We refer the readers to the two monographs [6,7] for more information on the history and results of the dominating set problem.

The concept of [1,2]-domination number was first investigated by Dejter [4]. Let S be a subset of V in a graph $G=(V,E)$, we define S to be a *[1,2]-set* if, for every vertex $v\in V\setminus S$, v is adjacent to at least one but no more than two vertices in S. The minimum cardinality of a [1,2]-set in G is called the *[1,2]-domination number*, denoted by $\gamma_{[1,2]}(G)$. A [1,2]-set with

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cardinality $\gamma_{[1,\,2]}(G)$ is called a $\gamma_{[1,\,2]}(G)$-set. Obviously, if S is a [1,2]-set, then it is also a dominating set, but a dominating set may not be a [1,2]-set, which implies $\gamma_{[1,\,2]}(G) \geq \gamma(G)$.

In [2], Chellali et al. studied [1,2]-domination numbers of graphs and gave some bounds for $\gamma_{[1,\,2]}(G)$. At the end of their paper, a number of open problems are proposed, some of which have already been solved by Yang et al. [9]. We state one of the problems as follows.

Problem 1 (Question 8 of [2]). For which graphs G is $\gamma(G) = \gamma_{[1,2]}(G)$?

The generalized Petersen graph, received this name by Watkins [8], is a generalization of the classic Petersen graph. (The concept was first defined by Coxeter [3] but not with that name). For each n and k ($n > 2k$), the *generalized Petersen graph* $P(n, k)$ is a graph on $2n$ vertices, where the vertex set is the union of $U = \{u_1, u_2, \ldots, u_n\}$ and $V = \{v_1, v_2, \ldots, v_n\}$, and the edge set is $\{u_iu_{i+1}, u_iv_i, v_iv_{i+k} : 1 \leq i \leq n, \text{subscripts modulo } n\}$. Observe that the classic Petersen graph is exactly $P(5, 2)$.

Ebrahimi et al. [5] determined the exact values of the domination numbers of three classes of generalized Petersen graphs.

Theorem 1 [5]. *If $n \geq 3$, then we have*

$$\gamma(P(n,1)) = \begin{cases} \frac{n}{2}+1, & \text{if } n \equiv 2 \pmod 4, \\ \lceil \frac{n}{2} \rceil, & \text{otherwise,} \end{cases}$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

Theorem 2 [5]. *For $n \geq 5$ we have $\gamma(P(n,2)) = \lceil \frac{3n}{5} \rceil$.*

Theorem 3 [5]. *For $n \geq 7$ we have*

$$\gamma(P(n,3)) = \begin{cases} \frac{n}{2}+1, & \text{if } n \equiv 2 \pmod 4, \\ \lceil \frac{n}{2} \rceil, & \text{if } n \equiv 0, 1 \pmod 4 \text{ or } n = 11, \\ \lceil \frac{n}{2} \rceil + 1, & \text{if } n \equiv 3 \pmod 4,\ n \neq 11. \end{cases}$$

In this paper, we focus on the [1,2]-domination numbers of these three classes of generalized Petersen graphs. In Section 2 we determine the exact values of $\gamma_{[1,\,2]}(P(n,\,1))$ and $\gamma_{[1,\,2]}(P(n,\,3))$, which are equal to the domination numbers of $P(n, 1)$ and $P(n, 3)$, respectively. In Section 3 we present the exact value of $\gamma_{[1,\,2]}(P(n,\,2))$. Finally, we conclude the paper with some remarks.

2. [1,2]-domination numbers of *P*(*n*, 1) and *P*(*n*, 3)

In this section, we establish the exact values of the [1,2]-domination numbers of $P(n, 1)$ and $P(n, 3)$, repectively.

Theorem 4. *If $n \geq 3$, then* $\gamma_{[1,2]}(P(n,1)) = \begin{cases} \frac{n}{2}+1, & \text{if } n \equiv 2 \pmod 4, \\ \lceil \frac{n}{2} \rceil, & \text{otherwise.} \end{cases}$

Proof. First, we will construct a [1,2]-set S of $P(n, 1)$. We use two lines: in the first line there are values of vertices $\{u_1, u_2, \ldots, u_n\}$, and in the second line there are values of vertices $\{v_1, v_2, \ldots, v_n\}$, where u_i lies above v_i for all i, $1 \leq i \leq n$. If the value of u_i is 1, then $u_i \in S$; if the value of v_i is 1, then $v_i \in S$.

(1) $n \equiv 0 \pmod 4$:
1000 1000 … 1000
0010 0010 … 0010

(2) $n \equiv 1 \pmod 4$:
1000 1000…1000 1
0010 0010…0010 0

(3) $n \equiv 2 \pmod 4$:
1000 1000 … 1000 1001 00
0010 0010 … 0010 0100 10

(4) $n \equiv 3 \pmod 4$:
1000 1000 … 1000 100
0010 0010 … 0010 001

By the construction, one can easily check that S is indeed a [1,2]-set. Therefore,

$$\gamma_{[1,2]}(P(n,1)) \leq \begin{cases} \frac{n}{2}+1, & \text{if } n \equiv 2 \pmod 4, \\ \lceil \frac{n}{2} \rceil, & \text{otherwise.} \end{cases}$$