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Inverse spectral problems for discontinuous Sturm–Liouville problems of Atkinson type☆

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ABSTRACT

We investigate inverse spectral problems for discontinuous Sturm–Liouville problems of Atkinson type whose spectrum consists of a finite set of eigenvalues. For given two finite sets of interlacing real numbers, there exists a class of Sturm–Liouville equations such that the two sets of numbers are exactly the eigenvalues of their associated Sturm–Liouville problems with two different separated boundary conditions. The main approach is to give an equivalent relation between Sturm–Liouville problems of Atkinson type and matrix eigenvalue problems, and the theory of inverse matrix eigenvalue problems.

1. Introduction

It is well-known that the spectrum of standard self-adjoint Sturm–Liouville problem (SLP) is infinite (see [20]). However, Atkinson in [1] indicated that the SLPs may have finite eigenvalues under the assumption that the coefficient functions satisfy some conditions. For any given integer n, a class of SLPs with exactly n eigenvalues was constructed in paper [2]. In 2009, Kong et al. in [3] showed that any SLP of Atkinson type with separated boundary conditions is equivalent to a matrix eigenvalue problem of the form of $HX = \lambda WX$ in the sense that the two problems have exactly the same eigenvalues. The results in [2,3] showed that the SLPs with finite spectrum can be transferred to the matrix eigenvalue problems and vice versa. As an application of [2,3], the inverse SLPs were studied in [4,5]. In paper [4], Volkmer and Zettl studied an inverse spectral problems for SLPs with Dirichlet boundary conditions. Later, Kong and Zettl promoted it to the general separated boundary conditions. Precisely, for given two finite sets of interlacing real numbers, there exists a class of second order differential equations such that the two sets of numbers are exactly the eigenvalues of their associated SLPs with two different separated boundary conditions.

There is no surprise that a large number of researchers pay attention to the discontinuous SLP with inner discontinuous points since these problems are of widely applications with discontinuous material properties, such as heat and mass transfer, vibrating string problems when the string loads with additional point masses, diffraction problems and so on (see [6–14]). In 2009, Ao et al. in [17] showed that the discontinuous SLP of Atkinson type also have finite spectrum and gave the matrix representations of SLPs with transmission conditions in [18]. Inspired by above mentioned papers, in this paper, we will study the inverse SLP with transmission conditions as follows:

$$-(p(x)y'(x))' + q(x)y(x) = \lambda w(x)y(x), \quad x \in [0, \xi) \cup (\xi, 1], \tag{1.1}$$

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subject to the separated boundary conditions (BCs) at the endpoints 0 and 1

$$\cos\alpha y(0) - \sin\alpha (py')(0) = 0, \quad 0 \le \alpha < \pi, \tag{1.2}$$

$$\cos\beta y(1) - \sin\beta (py')(1) = 0, \quad 0 < \beta \le \pi, \tag{1.3}$$

the transmission conditions at $x = \xi$ are

$$y(\xi + 0) - \delta_1 y(\xi - 0) - \delta_2 (py')(\xi - 0) = 0, \tag{1.4}$$

$$(py')(\xi + 0) - \delta_3 y(\xi - 0) - \delta_4 (py')(\xi - 0) = 0, \tag{1.5}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter, $\frac{1}{p(x)}$, $q(x)$, $w(x) \in L([0, \xi) \cup (\xi, 1])$ (here $L([0, \xi)\cup(\xi, 1])$ denotes the set of real-valued integrable functions on $[0, \xi)\cup(\xi, 1]$) and $q(x)$ have finite limits $q(\xi \pm 0) = \lim_{x\to(\xi\pm 0)} q(x)$, δ_1, δ_3 and δ_4 are non-zero numbers with $\delta_1\delta_4 - \delta_2\delta_3 = \rho > 0$.

The Sturm-Liouville equation (1.1) is said to be of *Atkinson type* if for two positive integers $m \ge 1$, $n \ge 1$, there exists a partition of the interval $[0, \xi)\cup(\xi, 1]$

$$0 = a_0 < a_1 < a_2 < \cdots < a_{2m+1} = \xi = b_0 < b_1 < b_2 < \cdots < b_{2n+1} = 1, \tag{1.6}$$

such that

$$\begin{aligned} &r(x) = \frac{1}{p(x)} = 0, \quad x \in [a_{2i}, a_{2i+1}] \cup [a_{2m}, a_{2m+1}) \cup (b_0, b_1] \cup [b_{2j}, b_{2j+1}], \quad i = 0, 1, 2, \dots, m-1, \quad j = 1, 2, \dots, n, \\ &\int_{a_{2i+1}}^{a_{2i+2}} r(x)dx > 0, \quad i = 0, 1, 2, \dots, m-1, \quad \int_{b_{2j+1}}^{b_{2j+2}} r(x)dx > 0, \quad j = 0, 1, 2, \dots, n-1; \end{aligned} \tag{1.7}$$

$$q(x) = 0, \; x \in [a_{2i+1}, a_{2i+2}] \cup [b_{2j+1}, b_{2j+2}], \quad i = 0, 1, 2, \dots, m-1, \quad j = 0, 1, 2, \dots, n-1; \tag{1.8}$$

$$\begin{aligned} &w(x) = 0, \; x \in [a_{2i+1}, a_{2i+2}] \cup [b_{2j+1}, b_{2j+2}], \; i = 0, 1, 2, \dots, m-1, \; j = 0, 1, 2, \dots, n-1, \\ &\int_{a_{2i}}^{a_{2i+1}} w(x)dx > 0, \quad i = 0, 1, 2, \dots, m, \quad \int_{b_{2j}}^{b_{2j+1}} w(x)dx > 0, \quad j = 0, 1, 2, \dots, n. \end{aligned} \tag{1.9}$$

If Eq. (1.1) is of Atkinson type and BCs (1.2) and (1.3) is self-adjoint, we say SLP (1.1)–(1.5) is of *Atkinson type*.

In this paper, using the equivalence between SLP with transmission conditions and matrix eigenvalue problem, we investigate the inverse spectral problems for discontinuous SLPs of Atkinson type, i.e. for given two positive integers m and n, and two finite sets of interlacing real numbers, we can construct a family of SLP whose spectrum entirely consists of two advanced sets of numbers. In fact, when (1.7)–(1.9) hold, the number of eigenvalues of (1.1)–(1.5) with $\delta_2 \neq 0$ is either $m + n + 2$, $m + n + 1$, or $m + n$; while for $\delta_2 = 0$, the number of eigenvalues of (1.1)–(1.5) is either $m + n + 1$, $m + n$, or $m + n - 1$(where m and n are related to the partition of $[0, \xi)\cup(\xi, 1]$) (see [17] for details).

This paper is organized as follows: we give the main theorems in Section 2, the relationship between given two finite sets of interlacing real numbers and matrix eigenvalue problems is discussed in Section 3, the proofs of the main theorems are given in Section 4.

2. Main results

Definition 2.1. A discontinuous SLP of Atkinson type is said to be equivalent to a matrix eigenvalue problem if both of them have exactly the same eigenvalues.

For convenience, we notate

$$\begin{aligned} &p_i = \left(\int_{a_{2i-1}}^{a_{2i}} r\right)^{-1}, \quad i = 1, 2, \dots, m, \\ &q_i = \int_{a_{2i}}^{a_{2i+1}} q, \quad w_i = \int_{a_{2i}}^{a_{2i+1}} w, \quad i = 0, 1, 2, \dots, m, \\ &\widetilde{p}_j = \left(\int_{b_{2j-1}}^{b_{2j}} r\right)^{-1}, \quad j = 1, 2, \dots, n, \\ &\widetilde{q}_j = \int_{b_{2j}}^{b_{2j+1}} q, \quad \widetilde{w}_j = \int_{b_{2j}}^{b_{2j+1}} w, \quad j = 0, 1, 2, \dots, n. \end{aligned} \tag{2.1}$$