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A node-centered finite volume method for a fracture model on triangulations ☆

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ABSTRACT

In this paper, a node-centered finite volume method based on triangulations for a fracture model is presented, in which we restrict the pressure to the linear finite element space while the velocity can be approximated by constant vectors element by element. The numerical scheme is established just associated with the pressure to avoid the saddle-point problem. Error estimates of $O(h)$ accuracy for the discrete H^1 semi-norm and the discrete L^2 norm of pressure p and the $(L^2)^2$ norm of velocity $\mathbf{u}$ are developed on general triangulations. Under an additional assumption about essentially symmetric control volumes, the error estimates for the pressure p can be improved to $O(h^{3/2})$. Finally, numerical experiments are carried out to verify the accuracy and convergence rates for the proposed finite volume scheme.

1. Introduction

In this paper, we consider the problem of a single phase fluid in a domain containing fractures. Since the fracture is supposed to have a small thickness with respect to the width of the domain, it is regarded as an $(n-1)$-dimensional interface in the n-dimensional porous medium. However, the fracture permeability may vary by several orders of magnitude, thus the fracture has a significant influence on the behavior of fluid flow. In general, there are two kinds of fractures. Fractures with higher permeability than the surrounding domains act as fast pathways, and those with lower permeability act as geological barriers. For the higher permeability case, the fluid tends to flow into fractures and then along them, so the velocity is not supposed to be continuous across fractures while for the lower permeability case, the fluid has a tendency to avoid fractures, so the pressure is not identical on both sides of fractures. The problem of fluid flow in fractured porous medium has been concerned by many researchers. A fracture model under the assumption of discontinuous velocity was proposed in [3]. And a model based on Robin boundary conditions at the fracture was introduced in [5]. In [22], a series of fracture models related to a parameter ξ were discussed, and they generalized the earlier problems to handle fractures with higher and lower permeability than that in the surrounding domains. It is the fracture model that we will consider in this paper, in which the fracture is modeled individually, and the pressure and the normal component of the velocity are all discontinuous across the interface. The flow is governed by Darcy's law together with the law of mass conservation in the fracture and surrounding domains, and the interaction between the fracture and surrounding matrix is taken into account. Similar researches for the fluid flow in fractured porous medium were developed in [4,15,20,23]. And we also

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introduce some numerical methods for fracture models, such as an extended finite element method in [16], the mixed finite element with domain decomposition method in [18], and a block-centered finite difference method in [21]. The finite volume method (FVM) as a discretization technique is widely used for numerical simulations in engineering fields due to its local conservative property and flexibility. The basic idea of FVM is to integrate equations of conservation laws in control volumes and by the divergence formula, the integrals in volumes are transformed to integrals of fluxes over boundaries which can be approximated with respect to unknowns. Different constructions of control volumes and different expressions of fluxes make different FVMs. The two point flux approximation finite volume method (TPFA) [10,12,13,17] is a cell-centered scheme, and each flux is computed by using only two unknowns defined on each side of any edge. In [17], a 4-point finite volume scheme was proposed based on this idea for a diffusion-convection problem on triangular meshes. The TPFA solution is piecewise constant and it is very cheap to implement, but meshes of TPFA are assumed to satisfy the orthogonal conditions described in [10]. In order to eliminate orthogonal conditions, one idea is that we use additional values to compute fluxes. The multi-point flux approximation scheme (MPFA) developed in [1,2,11] is based on such a construction. The MPFA approximation is linear inside sub-cells around every vertex by introducing additional unknowns defined on edges. And edge unknowns can be eliminated by conservativity equations, so MPFA is also a numerical scheme in terms of cell unknowns. The finite volume method proposed in this paper is a node-centered scheme. Compared to TPFA (or MPFA), the big difference of this method is that unknowns are defined on vertices instead of in cells. The construction of dual partition is similar to the classical Box method in [6,24]. For any given triangular mesh, control volumes are constructed by connecting the circumcenters of triangles and medians of edges, which forms a one-to-one correspondence with all vertices. The numerical solution is restricted to conforming linear finite element spaces. This kind of finite volume method was applied to general elliptic boundary value problems in [7,8], and we also refer to Refs. [9,14,19] for more details about finite volume methods.

In this paper, a node-centered finite volume method based on triangulations is considered for the coupled fracture model. Inspired by TPFA, we express the flux in every edge by only two unknowns defined on vertices, hence it preserves the property of TPFA, very easy to implement. For the primal partition, we assume that triangular meshes of surrounding domains align in the fracture to form a matching partition for the fracture. The degrees of freedom for the pressure are defined on vertices of the primal partition in the fracture and surrounding domains. Compared to the mixed finite element method in [22], the numerical scheme proposed in this paper is only related to the pressure and then the velocity is calculated element by element, which enables us to avoid the saddle-point problems resulting from the mixed finite element method. The discrete H^1 semi-norm and the L^2 norm for the pressure p are defined, and error estimates show that they all hold first-order accuracy for any general triangular meshes. We also establish $O(h)$ estimate for the error of the $(L^2)^2$ norm of the velocity $\mathbf{u}$. Furthermore, by the definition of the essentially symmetric control volumes introduced in [7], we form symmetric control volumes except for two strings volumes lying on both sides of the fracture, and in this case, error estimates for pressure p can be improved to $O(h^{3/2})$. Three numerical examples are carried out to verify the convergence rates of the finite volume scheme for the model with higher, lower or even anisotropic permeability in the fracture. Hence, the accuracy of the node-centered finite volume method for the fracture model is proved theoretically and demonstrated numerically in our paper. Besides, the principle of FVM is integrating a partial differential equation over control volumes, so an important feature of FVM is the local conservation property. We compare the node-centered finite volume method with the corresponding finite element method (FEM) in numerical experiments, which shows that the finite volume scheme holds local mass conservation.

The rest of the paper is organized as follows. In Section 2, we briefly describe the fracture model and give some notations. In Section 3, a node-centered finite volume method is established, and the well-posedness of the numerical scheme is analyzed. In Section 4, error estimates and convergence rates for the discrete H^1 and the discrete L^2 norm of p and the $(L^2)^2$ norm of $\mathbf{u}$ are proved. In Section 5, numerical experiments are conducted to test the accuracy and efficiency of the proposed finite volume method. A numerical comparison between FVM and FEM is developed, and the node-centered finite volume scheme shows good local mass conservation.

2. Description of the problem and notation

We consider the fracture model described in [22], which couples 2-dimensional elliptic equations in surrounding domains with a 1-dimensional elliptic equation in the fracture. For simplicity, the model with a vertical fracture in the domain is discussed, and the proposed node-centered finite volume method with triangular meshes can be extended to complex domains with non-vertical fractures. Let $\Omega = [a_1, a_2] \times [b_1, b_2]$ be a rectangular domain in $\mathbb{R}^2$ with the boundary Γ and we suppose that $\gamma = \{x = x_f\} \times [b_1, b_2] \subset \Omega$ is a one-dimensional surface. Ω is separated by γ into two bounded subdomains:

$$\Omega_1 = [a_1, x_f) \times [b_1, b_2], \text{ and } \Omega_2 = (x_f, a_2] \times [b_1, b_2],$$

$$\Omega \subset \mathbb{R}^2, \ \Omega \backslash \gamma = \Omega_1 \cup \Omega_2, \ \Gamma = \partial \Omega.$$

Denote by Γ_i the part of the boundary of Ω_i in common with the boundary of Ω, $\Gamma_i = \partial\Omega_i \cap \Gamma$, $i = 1, 2$. The exact velocity $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_f)$ and pressure $p = (p_1, p_2, p_f)$ are defined on Ω_1, Ω_2 and γ, respectively. We consider the following fracture problem, in which the flow in both the fracture and the surrounding matrix is governed by Darcy's law together