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Finite difference scheme for one system of nonlinear partial integro-differential equations

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ABSTRACT

System of Maxwell equations is considered. Reduction to the integro-differential form is given. Existence, uniqueness and large time behavior of solutions of the initial-boundary value problem for integro-differential model with two-component and one-dimensional case are studied. Finite difference scheme is investigated. Wider class of nonlinearity is studied than one has been investigated before. FreeFem++ realization code and results of numerical experiments are given.

1. Introduction

Integro-differential equations arising in mathematics and physics often contain derivatives with respect to several variables; therefore, these equations are referred to as partial integro-differential equations. Numerous publications deal with the study of integro-differential equations of various kinds (see the bibliography in [9,20] and references therein). Many scientific papers are dedicated to the investigation of nonlinear integro-differential equations of the parabolic type. Such an integro-differential model appears, for example, in the mathematical modeling of penetration of electromagnetic field into a medium whose electric conductivity substantially depends on temperature. On the basis of the Maxwell system of differential equations [23,31,34] a general statement of the above-mentioned diffusion problem was given in [7,8]. The problem also was reduced to an integro-differential model. The corresponding initial-boundary value problems were posed. Uniqueness and existence of their global solutions were considered.

Let us describe briefly reduction to the above-mentioned integro-differential model. Assume that the massive body is placed in a variable magnetic field. It is necessary to describe the field distribution inside the body. According to Sedov [34] consider the following system of Maxwell equations describing the interaction of electromagnetic field with medium:

$$-\frac{1}{c}\frac{\partial H}{\partial t} = rot\, E, \tag{1.1}$$

$$div\, H = 0, \tag{1.2}$$

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$$\frac{4\pi}{c}J = rot\,H, \tag{1.3}$$

$$J = \sigma E, \tag{1.4}$$

where $E = (E_1, E_2, E_3)$ and $H = (H_1, H_2, H_3)$ are vectors of electric and magnetic fields correspondingly, σ is conductivity of the medium, c is speed of light in vacuum and J is a vector of current density. Introducing the resistance $\rho = \frac{1}{\sigma}$ from (1.3) and (1.4) we get

$$E = \rho\frac{c}{4\pi}rot\,H.$$

Substituting this relation in (1.1) we arrive at

$$\frac{\partial H}{\partial t} + \frac{c^2}{4\pi}rot\,(\rho\,rot\,H) = 0. \tag{1.5}$$

Penetrating into a substance, a variable magnetic field induces variable electric field which causes appearance of currents. Currents lead to heating matter and raising its temperature θ that influences on the resistance ρ. According to the discussion given in [31], it follows that the power-like change of the temperature leads to a change of the resistance ρ on several orders. Therefore, for the large variations of the temperature the dependence $\rho = \rho(\theta)$ must be taken into account. The last significant restriction that should be made is due to the assumption that the change of the temperature of substance under the influence of the current J obeys Joule–Lenz law, which has the form

$$c_v\frac{\partial\theta}{\partial t} = \rho J^2. \tag{1.6}$$

Here ρ is the density of medium, c_v is heat capacity. In general, they also are depended on the temperature θ. Eq. (1.6) does not consider the process of heat transfer by heat conductivity and radiation. There is not also considered a number of other physical effects. However, in this form, the system (1.5), (1.6) is rather complicated from a mathematical point of view. Many scientific papers are dedicated to the investigation and numerical solution of system of Maxwell equations and to models like that equations (see, for example, [20,30] and references therein). One must also note that (1.5), (1.6) type models arise in mathematical modeling of many other processes as well (see, for example, [4,20] and references therein).

Eq. (1.6) can be rewritten in the following form

$$\frac{c_v(\theta)}{\rho(\theta)}\frac{\partial\theta}{\partial t} = J^2.$$

Introducing the function

$$S(\theta) = \int_{\theta_0}^{\theta}\frac{c_v(\xi)}{\rho(\xi)}d\xi,$$

we have

$$\frac{\partial S}{\partial t} = J^2.$$

Suppose that process starts at $t = 0$ moment, which corresponds to a constant temperature over substance θ_0. Integrating this equation over the interval $[0, t]$, we obtain

$$S(\theta(x,t)) = \int_0^t J^2 d\tau.$$

Functions c_v and ρ are positive in the physical sense, therefore $S(\theta)$ is monotonically increasing function. Thus, there exists uniquely defined inverse function $\theta = \varphi(S)$, related to function $S(\theta)$ by the following relation $\varphi(S(\theta)) = \theta$. So, we can write

$$\theta(x,t) = \varphi\left(\int_0^t J^2 d\tau\right).$$

From (1.3) we have

$$J = \frac{c}{4\pi}rot\,H,$$

and so,

$$\theta(x,t) = \varphi\left(\int_0^t \left|\frac{c}{4\pi}rot\,H\right|^2 d\tau\right).$$