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Numerical solution of nonlinear fractional integro-differential equations with weakly singular kernels via a modification of hat functions

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ABSTRACT

In the present paper, a modification of hat functions (MHFs) has been considered for solving a class of nonlinear fractional integro-differential equations with weakly singular kernels, numerically. The fractional order operational matrix of integration is introduced. We provide an error estimation for the approximation of a function by a series of MHFs. To suggest a numerical method, the main problem is converted to an equivalent Volterra integral equation of the second kind and operational matrices of MHFs are used to reduce the problem to the solution of bivariate polynomial equations. Finally, illustrative examples are provided to confirm the accuracy and validity of the proposed method.

1. Introduction

It has been revealed that many phenomena in several branches of science and engineering can be modeled using fractional calculus [1,2]. Among these phenomena, a diversity of physical problems have been formulated as fractional order integro-differential equations with weakly singular kernels. This kind of equations appears in the field of elasticity and fracture mechanics [3], radiative equilibrium [4], heat conduction problem [5], etc. Fractional order equations also appear in fluid dynamics, where the fractional Basset-type equation describes the unsteady motion of a sphere immerse in a Stokes fluid [6]. The equations in Section 6 are generalizations of this equation. In most cases, obtaining an analytical solution of fractional differential, integral and integro-differential equations is impossible or inconvenient. Consequently, introducing numerical algorithms to get high accuracy solutions is very important. Therefore, various numerical methods have been developed by many researchers for obtaining approximate solutions of this kind of equations. The most frequently used methods are Adomian Decomposition Method [7,8], Variational Iteration Method [9,10], Differential Transform Method [11,12] and Spectral Method [13,14].

In this work, we consider the following nonlinear fractional integro-differential equation with weakly singular kernel:

$$ {}_0^C D_t^{\alpha} y(t) = g(t) + p(t)y(t) + \int_0^t (t-s)^{-\beta} y^m(s)ds, \quad \alpha > 0, \ 0 \le \beta < 1, \ t \in I(T), \tag{1} $$

$$ y^{(i)}(0) = y_0^{(i)}, \quad i = 0, 1, \dots, \lceil \alpha \rceil - 1, \tag{2} $$

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where $y(t)$ is the unknown function to be determined, $g(t)$ and $p(t)$ are known continuous functions on $I(T):=[0, T]$, $y_0^{(i)}$ $(i = 0, 1, \dots, \lceil\alpha\rceil - 1)$ are given real numbers, $\lceil\alpha\rceil$ is the ceiling function of α, ${}_0^C D_t^\alpha$ is the Caputo fractional differential operator of order α, and m is a positive integer number.

Local and global existence and uniqueness results for the solution of fractional integro-differential equations have been obtained in [15] and [16], respectively. Collocation methods have been applied in [17] to the problem (1)–(2) in the case $m = 1$ (linear case) and the convergence of these methods has been analyzed. In [18], a spectral method based on the second kind Chebyshev polynomials has been considered to solve the linear case of (1). A hybrid collocation method has been suggested for solving fractional integro-differential equations in [19]. The CAS wavelet method has been proposed for solving a class of fractional integro-differential equation with a weakly singular kernel [20]. Yi et al. have presented the numerical solution of these equations using the Legendre wavelets method [21]. Also, the authors of [22] have proposed the second kind Chebyshev wavelets method to solve the fractional integro-differential equations with a weakly singular kernel.

In this paper, we introduce a numerical algorithm for the problem (1)–(2) in terms of MHFs. The present method uses some properties of the Caputo derivative and Riemann–Liouville integral operators to obtain an equivalent Volterra integral equation of the second kind. Using the operational matrix technique we reduce the original problem to the solution of a certain system of polynomial equations. In Section 2, we present some basic definitions and properties in fractional calculus. Section 3 is devoted to introducing the operational matrices of MHFs basis. In Section 4, an estimation of the error is given when a function is approximated in terms of MHFs. A numerical method is presented for solving the problem (1)–(2) in Section 5. Numerical examples are given in Section 6 to illustrate the applicability and accuracy of our method. Finally, the results of the paper are summarized in Section 7.

2. Basic concepts

In this section, we give some definitions and basic concepts which will be used further in this paper.

2.1. Fractional order integral and differential operators

There are a variety of definitions for fractional derivatives and integrals that have been used in the literature. In the present work, we consider the Caputo derivative ${}_0^C D_t^\alpha$ of order α and the Riemann–Liouville fractional integral operator I_t^α.

Let $\alpha \in \mathbb{R}$, $n - 1 < \alpha \le n$, $n \in \mathbb{N}$ and $y(t)$ be a real valued continuous function defined on $[0, \infty)$, then the Caputo fractional derivative of order $\alpha > 0$ is defined by [1]:

$$
{}_0^C D_t^\alpha y(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{d^n}{d\tau^n} y(\tau) d\tau, & n-1 < \alpha < n, \\ y^{(n)}(t), & \alpha = n, \end{cases}
$$

where $\Gamma(x)$ is the gamma function as

$$
\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.
$$

The Riemann–Liouville integral operator I_t^α of order $\alpha > 0$ is given by [1]:

$$
I_t^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} y(\tau) d\tau. \tag{3}
$$

For $\alpha, \beta > 0$, the Riemann–Liouville integral operator and the Caputo fractional derivative operator satisfy the following properties [1]:

$$
I_t^\alpha (I_t^\beta y(t)) = I_t^\beta (I_t^\alpha y(t)) = I_t^{\alpha+\beta} y(t),
$$

$$
{}_0^C D_t^\alpha ({}_0^C D_t^m y(t)) = {}_0^C D_t^m ({}_0^C D_t^\alpha y(t)) = {}_0^C D_t^{m+\alpha} y(t), \qquad m \in \mathbb{N},
$$

$$
{}_0^C D_t^\alpha (I_t^\alpha y(t)) = y(t),
$$

$$
I_t^\alpha ({}_0^C D_t^\alpha y(t)) = y(t) - \sum_{i=0}^{n-1} y^{(i)}(0) \frac{t^i}{i!}, \qquad n-1 < \alpha \le n, \quad t > 0. \tag{4}
$$

2.2. Properties of MHFs

Hat functions usually are defined on the interval [0, 1] and are continuous functions with shape hats [23]. Here we consider MHFs and extend the domain of definition to $[0, T]$. The interval is divided into n subintervals