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Numerical solution for system of Cauchy type singular integral equations with its error analysis in complex plane

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ABSTRACT

In this paper, the problem of finding numerical solution for a system of Cauchy type singular integral equations of first kind with index zero is considered. The analytic solution of such system is known. But it is of limited use as it is a nontrivial task to use it practically due to the presence of singularity in the known solution itself. Therefore, a residual based Galerkin method is proposed with Legendre polynomials as basis functions to find its numerical solution. The proposed method converts the system of Cauchy type singular integral equations into a system of linear algebraic equations which can be solved easily. Further, Hadamard conditions of well-posedness are established for system of Cauchy singular integral equations as well as for system of linear algebraic equations which is obtained as a result of approximation of system of singular integral equations with Cauchy kernel. The theoretical error bound is derived which can be used to obtain any desired accuracy in the approximate solution of system of Cauchy singular integral equations. The derived theoretical error bound is also validated with the help of numerical examples.

1. Introduction

System of Cauchy singular integral equations occur naturally in physics and engineering during the formulation of many boundary value problems containing different geometric singularities. Many crack problems in the field of fracture mechanics are formulated as system of Cauchy singular integral equations using Green's function [1]. For instance, in T-stress problem [2] near the tips of a cruciform crack with unequal arms the system of Cauchy type singular integral equations arise naturally. Also, it can be obtained by the decomposition of two dimensional Cauchy singular integral equation over a curve in complex plane [3]. Problems in the field of aerodynamics [4,5], queuing system analysis [6], electrocardiology [7], elasticity theory [8–10] are modeled as system of Cauchy singular integral equations.

In this paper, a numerical method to solve the system of Cauchy singular integral Eq. (1) is proposed

$$\fint_{-1}^{1}\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \dots & a_{NN} \end{pmatrix}\begin{pmatrix} v_1(t) \\ v_2(t) \\ \vdots \\ v_N(t) \end{pmatrix}\frac{1}{t-x}dt = \begin{pmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_N(x) \end{pmatrix}, \quad |x|< 1, \tag{1}$$

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with boundary conditions

$$v_j(t) = \begin{cases} 0 & \text{if } t = 1, \\ \text{unbounded} & \text{if } t = -1, \end{cases} \tag{2}$$

for $j = 1, 2, \ldots, N$.

In system (1), the coefficients $a_{ij};\ i,\ j = 1, 2, \ldots, N$, are known real constants, $g_j(x);\ j = 1, 2, \ldots, N$, and $v_j(t);\ j = 1, 2, \ldots, N$, are known and unknown complex valued functions defined on [-1,1], respectively. The singular integral appearing in each equation of system (1) is understood in the sense of Cauchy principal value (CPV) defined in Definition 2.4.

In system (1), each Cauchy equation is of index zero [11]. These equations are well studied [5,10–12]. They play a vital role in the study of mixed boundary value problems for partial differential equations [10] and have many applications in the field of aerodynamics [4], fracture mechanics [13], neutron transport [14] etc. Singular integral equations can also be used to determine the existence of travelling-wave solutions [15–17] of particular reaction-convection-diffusion equations [18].

The numerical methods developed for these kind of singular integral equations include: Galerkin method [19–22], collocation method [23,24], quadrature-collocation method [25], Nyström method [26], inverse method [27], real variable method [28], Sinc approximations [29]. However, the literature on numerical methods to find the solution of system of Cauchy type singular integral equations is still scarce. Although, the basic work related to this kind of system can be found in [10,30]. Bonis and Laurita [31] proposed a quadrature method for system of Cauchy type singular integral equations. Turhan [32] used Chebyshev polynomial based method for the solution of the system of Cauchy type singular integral equations of the first kind. The analytic solution (8) for one-dimensional Cauchy type singular integral (7) is well known [11]. We use this solution to derive the analytic solution for system (1) in Section 3. But still it is of limited use in practical situations. Since it is not possible to solve the singular integral on the right side of Eq. (9) for every choice of $\chi_j(t)$ due to the presence of singularity. Therefore, it is required to go for numerical solution for system (1). Hence we propose a residual based Galerkin method for solving the system (1) of Cauchy type singular integral equations. The error analysis of the proposed numerical method is also derived and validated through numerical examples.

2. Preliminaries

In this section, we discuss some of the important concepts [5,33] used in this work.

Definition 2.1. (**Legendre polynomial**) The n^{th} degree Legendre polynomials over the interval $[-1, 1]$ are defined as

$$p_n(t) = \frac{1}{2^n n!} \frac{d^n}{dt^n} (t^2 - 1)^n, \tag{3}$$

where n is a nonnegative integer.

Definition 2.2. (**Orthogonal property of Legendre polynomials**) The orthogonal property of Legendre polynomials over the interval $[-1, 1]$ makes them very useful and it can be stated as follows:

Let $p_m(t)$ and $p_n(t)$ be Legendre polynomials of degree m and n respectively, then we have

$$\int_{-1}^{1} p_m(t) p_n(t) dt = \begin{cases} 0 & \text{if } m \neq n, \\ \dfrac{2}{2n+1} & \text{if } m = n, \end{cases} \tag{4}$$

where m, n are nonnegative integers.

Definition 2.3. (**Hölder continuous function**) A function $\psi(t)$ is Hölder continuous if

$$| \psi(t) - \psi(x) | \leq c \, | t - x |^{\alpha}, \forall \ \ t, x \in D(\psi), \tag{5}$$

where c is a nonnegative real constant, α is an exponent of Hölder condition such that $0 < \alpha \leq 1$ and $D(\psi)$ denotes the domain of the function ψ.

Definition 2.4. (**Cauchy principal value[CPV]**) If $\psi(t) \in C^{0,\alpha}$, then

$$CPV \int_{-1}^{1} \frac{\psi(t)}{t - x} dt = \fint_{-1}^{1} \frac{\psi(t)}{t - x} dt = \lim_{\epsilon \to 0^+} \left(\int_{-1}^{x-\epsilon} \frac{\psi(t)}{t - x} dt + \int_{x+\epsilon}^{1} \frac{\psi(t)}{t - x} dt \right), \tag{6}$$

where $|x| \leq 1$ and $C^{0,\alpha}$ is the space of functions which are Hölder continuous on the interval $(-1, 1)$ with the exponent $0 < \alpha \leq 1$.

Definition 2.5. (**Hadamard conditions for well-posedness**) A problem is well-posed in the sense of Hadamard [34] if following conditions are satisfied:

1. it has a solution,
2. the solution is unique,
3. the solution depends continuously on given data.