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# On the wave interactions in the drift-flux equations of two-phase flows

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ABSTRACT

In the present work, we investigate the Riemann problem and interaction of weak shocks for the widely used isentropic drift-flux equations of two-phase flows. The complete structure of solution is analyzed and with the help of Rankine–Hugoniot jump condition and Lax entropy conditions we establish the existence and uniqueness condition for elementary waves. The explicit form of the shock waves, contact discontinuities and rarefaction waves are derived analytically. Within this respect, we develop an exact Riemann solver to present the complete solution structure. A necessary and sufficient condition for the existence of solution to the Riemann problem is derived and presented in terms of initial data. Furthermore, we present a necessary and sufficient condition on initial data which provides the information about the existence of a rarefaction wave or a shock wave for one or three family of waves. To validate the performance and the efficiency of the developed exact Riemann solver, a series of test problems selected from the open literature are presented and compared with independent numerical methods. Simulation results demonstrate that the present exact solver is capable of reproducing the complete wave propagation using the current drift-flux equations as the numerical resolution. The provided computations indicate that accurate results be accomplished efficiently and in a satisfactory agreement with the exact solution.



## 1. Introduction

One of the basic models of two-phase flow problems is drift-flux model which is widely used in wellbores, chemical and energy industries. This model was proposed and developed by Zuber and Findaly in [1] and employed by many authors over years. See for example [2–8] and references cited therein. Although the drift-flux model is one of the simplest models of two-phase flow problem and has some mathematical difficulties and theoretical drawbacks. Indeed, the system of equations is usually solved numerically [9,10] because of such mathematical difficulties due to interaction terms at the interface. However, many authors have been investigating the drift-flux model numerically. For instance, a linearized Roe-type Riemann solver based on quasilinear formulation of general drift-flux model is presented in [11]. A high resolution hybrid upwind scheme for isothermal drift-flux model describing gas-liquid flow in a long pipeline has been presented in [12]. The authors in [13] have also considered the approximate form of the Jacobian matrix for the drift-flux model and introduced a

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Roe-type Riemann solver for gas-liquid flow. In earlier, the analytical solution of the model equations is based on simplified assumption due to complicated nature of model equations. Indeed, the authors in [14] have presented an analytical solution for the homogeneous steady state flow through wellbore under isothermal assumptions. Recently, numerical simulation and analytical techniques to solve the Riemann problem for isothermal drift-flux two-phase flow model are discussed in [15]. The authors in [16] presented the existence of global solutions for Dirichlet problem in the context of viscous drift-flux model. In recent years, however, there is a noticeable increase in research related to the drift-flux model along with its fundamental issues such as the Riemann problem. The motivation of present work, therefore, is to study the existence and uniqueness of the elementary waves for the isentropic drift-flux two-phase equations and develop an analytical approach to solve the Riemann problem associated with such equations.

The solution to the Riemann problem becomes a very important topic in the field of hyperbolic system of conservation laws and it has attracted the attention of researchers in connection with numerical and analytical aspects of its solution. Although it is a special case of Cauchy problem with piecewise constant data having a single discontinuity, its solution constitutes the basic building blocks for the construction of solution to the general initial value problem using the random choice method proposed by Glimm [17]. Moreover, its solution consists of all the three elementary waves namely shocks and rarefaction waves along with contact discontinuities which enables the study of important properties of such elementary waves. In recent years, exact solution of the Riemann problem for the system of quasilinear hyperbolic PDEs have been extensively analyzed by the researchers and it is fully recognized that the Riemann problem is the most fundamental problem in the field of nonlinear hyperbolic conservation laws. The study of Riemann problem is a subject of great interest both from the mathematical and physical point of view due to its applications such as gasdynamics [18], multiphase mixtures [19], magnetogasdynamics [20], traffic flow problem [21], two-phase flows [22,23]. In [24], theory of solution for a strictly hyperbolic system of conservation laws under the assumption that each of the characteristic field is either genuinely nonlinear or linearly degenerate was established for solving the Riemann problem uniquely by considering the initial data sufficiently closed. Later, work was further extended to solve the Riemann problem for a class of hyperbolic systems with arbitrary initial data in [25]. In [26,27], author presented the solution of the Riemann problem for general system of conservation laws and obtained the uniqueness of weak solutions of Cauchy problem for general $2 \times 2$ conservation laws. An efficient solver for computing the exact solution of the Riemann problem for ideal and covolume gases was also presented in [28,29]. A different approach to present the exact Riemann solution for shallow water equations with bottom topography was presented in [30]. The Riemann problem and elementary wave interactions in isentropic magnetogasdynamics and shallow water equations have been discussed in [31,32] and for $3 \times 3$ magnetogasdynamics system without isentropic condition in [33,34]. Recently, authors in [35] studied the Riemann problem for van der Waals gas and discussed the effect of van der Waals parameter on the shock and rarefaction waves.

In this present work, we propose an analytical approach to solve the drift-flux model with Riemann initial data. This consists of solving conservation laws of both the phases and mixture momentum balance equation in one-dimensional infinite domain. Furthermore, the isentropic equation of state is assumed to construct the solution. We use the properties of shocks, rarefaction waves and contact discontinuities to reduce the PDEs to an algebraic system of equations. We solve this system of three algebraic equations using Newton's method with stopping criteria satisfying the error less than $10^{-10}$. We compare the exact solution with the numerical solution obtained by three different numerical schemes to validate the exact solver. Moreover, we study the interaction of weak shocks using the solution of the Riemann problem for the model equations as we shall see later.

The present work is organized as follows; in Section 2, we present the drift-flux equations governing two-phase flows. In Section 3, we show that system is strictly hyperbolic and establish the existence of shocks and rarefaction waves with discussion on the uniqueness conditions for both the shocks and rarefaction waves. In Section 4, we consider the Riemann problem with arbitrary initial data and show that it is solvable if the initial data satisfy a certain condition. In Section 5, we discuss the numerical schemes to validate the exact Riemann solver for the model equations. Three test cases with numerical validation are presented in Section 6. In Section 7, we discuss interaction of weak shocks of same family. Finally, conclusions and future work are presented in Section 8.

## 2. Governing equations

We consider the system of two-phase flow equations which is widely used in literature [11,15,36] and given by a system of PDEs coming from the balance laws of mass, momentum and energy. The one dimensional isentropic flows are given by two mass and momentum balance equations for each phase given by the following system of PDEs [36,37]:

$$\frac{\partial}{\partial t}(\alpha_1\varrho_1) + \frac{\partial}{\partial x}(\alpha_1\varrho_1 u_1) = 0, \tag{2.1a}$$

$$\frac{\partial}{\partial t}(\alpha_2\varrho_2) + \frac{\partial}{\partial x}(\alpha_2\varrho_2 u_2) = 0, \tag{2.1b}$$

$$\frac{\partial}{\partial t}(\alpha_1\varrho_1 u_1) + \frac{\partial}{\partial x}(\alpha_1\varrho_1 u_1^2) + \frac{\partial}{\partial x}(\alpha_1 p_1) - p^i\frac{\partial}{\partial x}(\alpha_1) = Q_1 + M_1^i, \tag{2.1c}$$