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Hybrid collocation methods for eigenvalue problem of a compact integral operator with weakly singular kernel

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A B S T R A C T

In this paper, we consider the hybrid collocation methods to solve the eigenvalue problem of a compact integral operator with weakly singular kernels of algebraic and logarithmic type. We obtain the global convergence rates for eigenvalues, the gap between the spectral subspaces and iterated eigenvectors. The numerical examples are presented to verify the theoretical estimates and also shown that this method is computationally useful in comparison to other methods.

1. Introduction

In the last decade, the eigenvalue problem of a compact integral operator with smooth kernel have been attracted much attention. For last some years, there has been considerable interest in solving the eigenvalue problem of an integral operator with weakly singular kernels. The integral equations with weakly singular kernels of algebraic and logarithmic type often arises as a reformulation of boundary value problems which cover many practical applications in mathematical physics.

Since the above problem cannot be solved exactly, the approximation methods are being used to solve the problem. The regularity properties of the solution for weakly singular integral equations of the second kind were studied by many authors ([15,16,18–21]). Chen et al. [7] developed fast collocation method for eigenvalue problem with weakly singular kernel using matrix compression techniques. Firstly, Rice [17] introduced nonlinear spline approximations for integral equations. To solve weakly singular integral equations, many authors use graded mesh techniques developed by Rice [17] instead of using uniform partition.

In the graded mesh techniques, the piecewise polynomials are used with a graded mesh on the interval according to the behavior of the function near the singular point. The disadvantage of using graded mesh techniques is that the subintervals near the singular point have very small length and it creates serious round-off errors. However to obtain more accurate solutions using graded mesh or piecewise polynomials, we need to refine the partition of the interval, then the size of the system of equations needs to be solved is very large and dense, which is computationally very expensive. In [14], Legendre Galerkin methods were discussed for both Fredholm integral equations of the second kind and the corresponding eigenvalue problem with weakly singular kernel, where the approximated space is global polynomial space instead of using piecewise polynomial space.

To avoid the difficulty arises in this direction, some more interesting results were also added in the literature. The Singularity preserving Galerkin method for Fredholm integral equations were developed by Cao and Xu [5]. In singularity preserving Galerkin method, the projected space is chosen as the direct sum of singular space and spline function space instead of

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using piecewise polynomial spaces or global polynomial spaces, where the basis for the singular space are nonpolynomial singular functions, which reflect the singularity of the exact eigen functions. This method uses quasiuniform partitions to avoid round-off errors and optimal order of convergence was obtained. However, the local convergence was obtained. To obtain global convergence hybrid collocation method for Volterra and Fredholm integral equations was developed in [4] and [3], respectively. In [9], the super-convergence rates have been obtained by using iterated Hybrid collocation method for Volterra integral equations with weakly singular kernel. The hybrid collocation method combines the idea of the singularity preserving method and graded collocation method and it avoids using small subintervals near zero. In this method the singular functions are approximated only on first and last subintervals, depending on the singularity arises on the end points and in other subintervals, functions are approximated by spline functions.

In this paper, we are interested in the approximations of eigenelements of the eigenvalue problem of a compact integral operator with weakly singular kernel using hybrid collocation methods. The global convergence also has been obtained for eigenvalues, eigenvectors and iterated eigenvectors. The theoretical error bounds are same as the error bounds obtained in singularity preserving collocation method. However, the actual advantage of this method has been shown in numerical results. The computational cost using the hybrid collocation methods are less in comparison to graded collocation and singularity preserving collocation method. The wavelet techniques also can be applied over this method to improve the result more.

We organize this paper as follows. In Section 2, we discuss the Hybrid collocation methods for the eigenvalue problem of a compact integral operator with algebraic and logarithmic kernel. In Section 3, we discuss the convergence rates for approximated eigenelements with exact eigenelements. In Section 4, we present numerical examples.

2. Hybrid collocation methods

Consider the following integral operator defined on the Banach space $\mathbb{X} = \mathcal{C}(\mathcal{J}),\ \mathcal{J} = [0, 1]$ by

$$\mathcal{K}u(s) = \int_0^1 k(s,t)u(t)\,\mathrm{d}t, \quad s \in [0,1], \tag{2.1}$$

where $k(s, t)$ is the kernel. The kernel is of the form $k(s,t) = m(s,t)g_\alpha(|s-t|)$, where $m(s,t) \in \mathcal{C}^{(n)}(\mathcal{J}\times\mathcal{J})$, $n \geq 0$ and

$$g_\alpha(|s-t|) = \begin{cases} |s-t|^{\alpha-1}, & \text{if } 0<\alpha<1, \\ \log|s-t|, & \text{if } \alpha = 1. \end{cases} \tag{2.2}$$

Then $\mathcal{K}$ is a compact linear integral operator from $\mathbb{X}$ to $\mathbb{X}$ (see, [11,12]).

We are interested to find the eigenvalue $\lambda \in \mathbb{C}\setminus\{0\}$ and the corresponding eigenvector $u \in \mathbb{X}$ for the following eigenvalue problem of the above compact linear integral operator $\mathcal{K}$:

$$\mathcal{K}u = \lambda u, \ \ \|u\| = 1. \tag{2.3}$$

Assume $\lambda \neq 0$ be the eigenvalue of $\mathcal{K}$ with algebraic multiplicity m and ascent ℓ. Let $\Gamma \subset \rho(\mathcal{K})$ be a simple closed rectifiable curve such that $\sigma(\mathcal{K}) \cap \mathit{int}\,\Gamma = \{\lambda\},\ 0 \notin \mathit{int}\,\Gamma$, where $\mathit{int}\,\Gamma$ denotes the interior of Γ.

As Hybrid collocation methods combines the idea of graded collocation methods with singularity preserving methods, lets have an introduction to both these methods with the convergence rates.

Graded collocation method: In the graded collocation method, the interval [0, 1] is divided into nonuniform partitions developed by Rice [17]. Let Π_k be the partition of [0, 1] defined by

$$x_{j,k} = \frac{b-a}{2}\left(\frac{2j}{k}\right)^q, \quad 0 \leq j \leq \frac{k}{2},$$
$$x_{j,k} = 1 - x_{k-j,k}, \quad \frac{k}{2} \leq j \leq k,$$

where $q = \frac{n}{\alpha+1}$, $n \geq 1$. Over this partition, we consider $\mathcal{S}_k^{n,\nu}$ the space of splines of order n on each $[x_{j,k}, x_{j+1,k}]$, $1 \leq j \leq k-1$ with ν continuity. In this method $\mathcal{S}_k^{n,\nu} = \mathcal{S}_k^n$ is the approximation space of $\mathbb{X}$. Now let $\mathcal{P}_n$ be the interpolatory projection from $\mathbb{X}$ into $\mathcal{S}_k^n$. Using the projection $\mathcal{P}_n$, the collocation method for the eigenvalue problem over the graded mesh of the interval [0, 1] is of the form

$$\mathcal{P}_n\mathcal{K}u_n = \lambda_n u_n,$$

where $u_n \in \mathcal{S}_k^n$ is the approximation of u. The iterated eigenvector is defined by $\tilde{u}_n = \frac{1}{\lambda_n}\mathcal{K}u_n$. For $u \in \mathcal{C}^n(\mathcal{J})$, the optimal order of convergence for the approximate solution u_n to u is

$$\|u-u_n\| = \mathcal{O}(k^{-n}), \quad |\lambda-\lambda_n| = \mathcal{O}(k^{-n-\alpha}),$$
$$\|u-\tilde{u}_n\| = \begin{cases} \mathcal{O}(k^{-n-\alpha}), & \text{if } 0<\alpha<1, \\ \mathcal{O}(k^{-n}\log(1/n)), & \text{if } \alpha = 1. \end{cases}$$

Singularity preserving method: In the singularity preserving method, the singularity expansion for the eigenvector has been evaluated, which are the generalization of the result of [8,16]. The singularity preserving Galerkin and collocation method