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Simultaneous inversion of the fractional order and the space-dependent source term for the time-fractional diffusion equation

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A B S T R A C T

In this paper, a simultaneous identification problem of the spacewise source term and the fractional order for a time-fractional diffusion equation is considered. Firstly, under some assumption and with two different kinds of observation data for one-dimensional and two-dimensional time-fractional diffusion equation, the unique results of the inverse problem are proven by the Laplace transformation method and analytic continuation technique. Then the inverse problems are transformed into Tikhonov type optimization problems, the existence of optimal solutions to the Tikhonov functional is proven. Finally, we adopt an alternating minimization algorithm to solve the optimization problems. The efficiency and stability of the inversion algorithm are tested by several one- and two-dimensional examples.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain with a smooth boundary $\partial\Omega$, we consider the following time-fractional diffusion equation

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = (Lu)(x,t) + F(x,t), x \in \Omega, \ t > 0, \tag{1}$$

with homogeneous Neumann boundary condition

$$\frac{\partial u}{\partial \mathbf{n}}(x,t) = 0, \ x \in \partial\Omega, \ t > 0, \tag{2}$$

and nonhomogeneous initial condition

$$u(x,0) = \varphi(x), x \in \overline{\Omega}, \tag{3}$$

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where $-L$ is a symmetric uniformly elliptic operator defined by

$$L(u) = \sum_{i=1}^{d} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{d} \theta_{i,j} \frac{\partial}{\partial x_j} u(x)\right) - c(x)u(x),$$

where $\theta_{i,j} \in C^1(\overline{\Omega}),\ \theta_{i,j} = \theta_{j,i}, c(x) \in C(\overline{\Omega}),\ c(x) \geq 0, \forall x \in \overline{\Omega}$, and there exists a constant $\nu > 0$ such that $\nu \sum_{i=1}^{d} \xi_i^2 \leq \sum_{i,j=1}^{d} \theta_{i,j}(x)\xi_i\xi_j,\ \forall x \in \overline{\Omega},\ \xi \in \mathbb{R}^d$. If $d = 1$, we take $\Omega = (0, l)$, and $x = 0, l$ are the boundary points of the interval [0, l]. If $d = 2$, $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$. $\frac{\partial u}{\partial \mathbf{n}}|_{\partial\Omega}$ represents the outward normal derivative along the boundary $\partial\Omega$. $\mathbf{n}$ is the outward unit normal vector of boundary $\partial\Omega$. Here, $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ is the Caputo fractional derivative defined by

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\eta)^{-\alpha} \frac{\partial u(x,\eta)}{\partial \eta} \mathrm{d}\eta,\ \ 0 < \alpha < 1,$$

where $\Gamma(\cdot)$ is a Gamma function.

The time-fractional diffusion equation has gained much attention among mathematicians during the last few decades, owing to its applicability of describing the anomalous diffusion processes well, especially the subdiffusion process, in which the mean squares variance grows slower than that in a Gaussian process for large time. There are extensive references on the theoretical analysis for time-fractional diffusion equations. In [1], the uniqueness of a classical solution was proved by a maximum principle. Luchko [2] gave the generalized solution to (1) with $F = 0$ with the help of the Mittag–Leffler function and proved the unique existence result. Sakamoto and Yamamoto [3] investigated comprehensively the well-posedness and the long-time asymptotic behavior of the solution for an initial boundary value problem. It shows that the spatial regularity of the solution is only moderately improved from that of the initial value, and the solution decays with order t^{-a} for large time. Gorenlfo et al. [4] proposed a definition of the Caputo derivative in the fractional Sobolev spaces and proved the maximal regularity of the solutions to some initial boundary value problems for the time-fractional diffusion equation with the Caputo derivative in the fractional Sobolev spaces.

Accompanied by the extensive researches of the time-fractional diffusion equation direct problem, the related inverse problems have attracted the interest of many researchers. Numerically, the backward problem for the time-fractional diffusion equation was considered by quasi-reversibility method, total variation regularization method, quasi-boundary value method, respectively in [5–7]. The uniqueness theories for identifying the space-dependent/time-dependent source term were established by Laplace transformation method, analytic continuation technique, analytic Fredholm theory, the maximum principle, respectively, in [8,9]. More recently, Wei et al. [10] proved the uniqueness and a stability estimate for the inverse time-dependent source problem from the boundary Cauchy data in a multidimensional time-fractional diffusion equation. Ruan and Wang [11] established conditional stability for an inverse problem of identifying a time-dependent source term for time fractional diffusion equation by using nonlocal measurement data and derived a priori convergence rate of regularized solutions by choosing a suitable mollification parameter. Jiang et al. [12] established a weak unique continuation property for time-fractional diffusion–advection equations by using the Laplace transform and the unique continuation properties for elliptic and parabolic equations, and proved the uniqueness of determining the spatial component in the source term by interior measurements. Liu and Zhang [13] established the multiple logarithmic stability of reconstructing the temporal component in the source term for time-fractional diffusion equations based on a reverse convolution inequality. Numerically, the inverse source problems were considered by quasi-boundary value method, reproducing kernel Hilbert space method, fixed point iteration method and Tikhonov regularization method, respectively, in [14–17]. For other inverse problems of the time-fractional diffusion equation, Li and Yamamoto [18] proved the uniqueness of determining the orders of multi-term time-fractional diffusion equation by the technique of eigenfunction expansion and Laplace transform. Jin and Rundell proposed an algorithm of the quasi-Newton type to recover a spatially varying potential coefficient in a time-fractional diffusion equation in [19]. Compared with the one function or one parameter identification problem, the researches of the simultaneous inversion problem for the time-fractional diffusion equation are not studied enough. Cheng et al. [20] established the uniqueness for identifying space-dependent diffusion coefficient and fractional order with Dirac delta function as an initial term by the Gelfand–Levitan theory. The uniqueness of simultaneously identifying the fractional order and the space-dependent diffusion coefficient in a one-dimensional time-fractional diffusion equation was proved by using boundary measurements in [21]. In [22], Sun and Wei proved the uniqueness of simultaneously identifying the fractional order and the zero order coefficient in one-dimensional time-fractional diffusion equation by the Laplace transformation and Gelfand–Levitan theory. Liu et al. [23] determined the initial status and heat flux simultaneously with the boundary observation data by using the Laplace transform and the unique extension technique in one-dimensional time-fractional diffusion equation. In [24], Ruan et al. proved the uniqueness of recovering the initial function and source term simultaneously by the method of eigenfunction expansion with two moment observation data.

We assume the source term $F(x, t)$ has the structure of temporal and spatial separation, i.e., $F(x,t) = p(t)f(x)$. In this paper, we consider the uniqueness for identifying fractional order α and source term $f(x)$ simultaneously for the cases of $d = 1$ and $d = 2$ by different observation data, that is, if $d = 1$, the observation data is

$$g(t) = u(0,t),\ \ t \in (0,T], \tag{4}$$