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Short Communication

An exponential stability criterion for nonlinear second-order functional differential equations with time-variable delays[☆]

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A B S T R A C T

This paper is concerned with the exponential stability for nonlinear second-order functional differential equations (FDEs) with time-variable delays. An exponential stability relationship between the FDEs and the corresponding ordinary differential equations (ODEs) is derived. It is proved under some appropriate conditions that the second-order FDEs can preserve the exponential stability of the corresponding ODEs. This stability result is also illustrated with a numerical approach.

1. Introduction

It is well-known that the second-order FDEs have been widely used to model many phenomena in physics, mechanics, biology and the other scientific fields. Hence, in the recent decades, the research for this type of equations has been paid great attention. In the existing research, stability of the equations is an important issue, whose fundamental results can be found in monographs [7,11,12] and the references therein. Based on the requests of actual problems, ones proposed various stability concepts and presented the corresponding stability results. In particular, for the linear second-order FDEs with constant coefficients p_i, q_i $(i = 1, 2)$ and delay $\tau > 0$:

$$x''(t) = p_1 x'(t) + p_2 x'(t-\tau) + q_1 x(t) + q_2 x(t-\tau), \quad t \geq 0, \tag{1.1}$$

Cahlon and Schmidt [8] and Yeniçerioğlu [14] studied asymptotic stability of the trivial solution with Pontryagin's theory of quasi-polynomials and gave a series of stability criteria. For the linear second-order FDEs with variable coefficients $p_i(t)$, $q_j(t)$ and delays $\theta_i(t), \tau_j(t) \geq 0$ $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m)$:

$$x''(t) + \sum_{i=1}^{n} p_i(t) x'(t - \theta_i(t)) + \sum_{j=1}^{m} q_j(t) x(t - \tau_j(t)) = f(t), \quad t \geq 0, \tag{1.2}$$

using the positivity of the Cauchy function, Bainov and Domoshnitsky [5] derived a sufficient condition of exponential stability, and Agarwal et al. [2] further improved their result and obtained the conclusion that Eq. (1.2) can preserve the

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exponential stability of the corresponding ODEs

$$x''(t)+\sum_{i=1}^{n}p_i(t)x'(t)+\sum_{j=1}^{m}q_j(t)x(t)=f(t),\quad t\geq 0 \tag{1.3}$$

whenever the delays are small enough. A related research for nonoscillation of FDEs (1.2) refers to Agarwal et al. monograph [1]. Moreover, for the linear second-order FDEs without damping terms, Berezansky et al. [6] and Domoshnitsky [10] also made some insights into exponential stability of the equations.

We note that the above research focus on the linear case of second-order FDEs. As to the nonlinear case, to the best of our knowledge, there are few papers involving this topic. In Mao [13], Zhang and Li [15] and Zhang and Niu [16], the authors investigated the exponential stability of first-order nonlinear FDEs, respectively. The presented approach can be considered as a nonlinear version of Azbelev's W-transform for the study of stability in Azbelev et al. [3] and Azbelev and Simonov [4]. Motivated by their ideas, in this paper, we extend the research to nonlinear second-order FDEs with time-variable delays and prove that the second-order FDEs can preserve the exponential stability of the corresponding ODEs under some appropriate conditions. The obtained result is also verified with a numerical illustration.

2. A class of nonlinear second-order FDEs with time-variable delays

This section will deal with the following nonlinear initial problems of second-order FDEs

$$\begin{aligned}&x''(t)=f\big(t,x(t),x(t-\alpha(t)),x'(t),x'(t-\beta(t))\big),\quad t\geq t_0;\\&x(t)=\varphi(t-t_0),\quad x'(t)=\varphi'(t-t_0),\quad t_0-\tau\leq t\leq t_0,\end{aligned} \tag{2.1}$$

where functions $\alpha(t),\beta(t):[t_0,+\infty)\to\mathbb{R}^+\cup\{0\}$ are continuous with $\tau_1:=\sup_{t\geq t_0}\alpha(t)<+\infty$, $\tau_2:=\sup_{t\geq t_0}\beta(t)<+\infty$ and $\tau=\max\{\tau_1,\tau_2\}$, $\varphi:[-\tau,0]\to\mathbb{R}^n$ is continuous and differentiable, and $f:[t_0,+\infty)\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ is continuously differentiable with respect to variable t and satisfies Lipschitz condition with constants $L_i>0$:

$$\big\|f(t,x_1,x_2,x_3,x_4)-f(t,\tilde{x}_1,\tilde{x}_2,\tilde{x}_3,\tilde{x}_4)\big\|\leq\sum_{i=1}^{4}L_i\|x_i-\tilde{x}_i\|,\quad\forall t\in[t_0,+\infty),\quad x_i,\tilde{x}_i\in\mathbb{R}^n\ (i=1,2,3,4), \tag{2.2}$$

where $\|\cdot\|$ denotes a given norm in $\mathbb{R}^n$. In order to assure that Eq. (2.1) can admit a trivial solution $x(t;\, t_0,\, \varphi,\, \varphi')\equiv 0$ when $\varphi(t)=0$ for all $t\in[t_0-\tau,t_0]$, we always assume that $f(t,0,0,0,0)\equiv 0$ for all $t\geq t_0$.

Let $\mathfrak{C}:=\mathbb{C}([-\tau,0],\ \mathbb{R}^{2n})$ be the Banach space consisting of continuous functions mapping the interval $[-\tau,0]$ into $\mathbb{R}^{2n}$ with norm $\|\phi\|=\sup_{\theta\in[-\tau,0]}\max_{i=1,2}\{\|\phi_i(\theta)\|\}$, where $\phi=\begin{bmatrix}\phi_1\\\phi_2\end{bmatrix}\in\mathfrak{C}$ with $\phi_1,\phi_2\in\mathbb{C}([-\tau,0],\ \mathbb{R}^n)$. Write

$$\hat{x}(t)=x'(t),\quad X(t)=\begin{bmatrix}x(t)\\\hat{x}(t)\end{bmatrix},\quad\Phi(t)=\begin{bmatrix}\varphi(t)\\\varphi'(t)\end{bmatrix},\quad F(t,X_t)=\begin{bmatrix}\hat{x}(t)\\f\big(t,x(t),x(t-\alpha(t)),\hat{x}(t),\hat{x}(t-\beta(t))\big)\end{bmatrix},$$

where $X_t\in\mathfrak{C}$ is defined by $X_t(\theta)=X(t+\theta)$ for all $\theta\in[-\tau,0]$. With these settings, problem (2.1) can be rewritten as

$$X'(t)=F(t,X_t),\quad t\geq t_0;\quad X_{t_0}=\Phi(t-t_0). \tag{2.3}$$

Applying Theorem 2.3 in Hale [11, Chap.2] to (2.3), we conclude that problem (2.1) has a unique solution $x(t;\, t_0,\, \varphi,\, \varphi')$ on $[t_0-\tau,+\infty)$ whenever the Lipschitz condition (2.2) holds.

3. An exponential stability criterion

In this section, we will deal with the exponential stability of problem (2.1). A stability relation between delay problem (2.1) and the corresponding ODE problem:

$$y''(t)=f\big(t,y(t),y(t),y'(t),y'(t)\big),\quad t\geq t_0;\quad y(t_0)=y_0\equiv\varphi(0),\quad y'(t_0)=y_0'\equiv\varphi'(0) \tag{3.1}$$

will be given. For studying this issue, we introduce the following localized norm on space $\mathbb{C}([t_0-\tau,+\infty),\ \mathbb{R}^n)$:

$$\big\|x(t)\big\|_{\mathcal{N}}=\big\|x(t)\big\|+\big\|x'(t)\big\|,\quad\forall t\geq t_0-\tau.$$

Definition 3.1. A problem (2.1) is called exponentially stable if there exist constants μ, $\nu>0$ such that the solution of problem (2.1) satisfies

$$\big\|x(t)\big\|_{\mathcal{N}}\leq\mu\exp[-\nu(t-t_0)]\sup_{-\tau\leq s\leq 0}\|\varphi(s)\|_{\mathcal{N}},\quad\forall t\geq t_0. \tag{3.2}$$

Correspondingly, problem (3.1) is called exponentially stable if there exist constants μ, $\nu>0$ such that

$$\|y(t)\|_{\mathcal{N}}\leq\mu\exp[-\nu(t-t_0)]\|y_0\|_{\mathcal{N}},\quad\forall t\geq t_0. \tag{3.3}$$

An exponential stability relation between problems (2.1) and (3.1) can be stated as follows.