Applied Mathematics and Computation 328 (2018) 243–246

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Short Communication

Approximations to the solution of Cauchy problem for a linear evolution equation via the space shift operator (second-order equation example)

Ivan D. Remizov*

National Research University Higher School of Economics, 25/12 Bol. Pecherskaya Ulitsa, Room 224, Nizhny Novgorod, 603155, Russia

ARTICLE INFO

MSC:
35A35
35C99
35K15
35K30

Keywords:
Cauchy problem
Linear parabolic PDE
Approximate solution
Shift operator
Chernoff theorem
Numerical method

ABSTRACT

We present a general method of solving the Cauchy problem for a linear parabolic partial differential equation of evolution type with variable coefficients and demonstrate it on the equation with derivatives of orders two, one and zero. The method is based on the Chernoff approximation procedure applied to a specially constructed shift operator. It is proven that approximations converge uniformly to the exact solution.

1. Problem setting and approach proposed

Consider $x \in \mathbb{R}^1$, $t \geq 0$ and set the Cauchy problem for a second-order parabolic partial differential equation

$$\begin{cases} u'_t(t,x) = (a(x))^2 u''_{xx}(t,x) + b(x)u'_x(t,x) + c(x)u(t,x) = Hu(t,x), \\ u(0,x) = u_0(x). \end{cases} \tag{1}$$

The coefficients a, b, c, u_0 above are bounded, uniformly continuous functions $\mathbb{R}^1 \to \mathbb{R}^1$. This paper is dedicated to deriving of an explicit formula that expresses the solution of (1) in terms of a, b, c, u_0 assuming that the operator H is an infinitesimal generator of the C_0-semigroup $(e^{tH})_{t\geq 0}$. This assumption is a standard one in studies of evolution equations, the class of equations that considered equation belongs to. According to the general theory of C_0-semigroups [1] this assumption implies that the solution of the Cauchy problem (1) exists, is bounded and uniformly continuous with respect to x for each t, depends on u_0 continuously and can be represented in a form $u(t,x) = \left(e^{tH}u_0\right)(x)$. We apply the Chernoff theorem (see [1,2] or Theorem 1 below) to a specially constructed family of operators $(S(t))_{t\geq 0}$, and express e^{tH} in terms of a, b, c reaching the goal announced. We do not discuss the problem of finding the class of functions in which the solution is unique under certain assumptions on functions a, b, c, u_0, but keep in mind that e.g. for a heat equation there are known unbounded solutions.

The formula that provides the solution of (1) is given in Theorem 3.

* Corresponding author.
E-mail address: ivremizov@yandex.ru

https://doi.org/10.1016/j.amc.2018.01.057
0096-3003/

2. Technique employed

For linear operator H, the Chernoff theorem allows to reduce the problem of finding e^{tH} to the problem of finding an appropriate operator-valued function $S(t)$, which is called the Chernoff function, and then use the Chernoff formula $e^{tH} = \lim_{n\to\infty} S(t/n)^n$. One advantage of that step is that we can define $S(t)$ by an explicit formula that depends on the coefficients of the operator H. Members of Smolyanov's group employed this technique using integral operators as Chernoff functions to find solutions to parabolic equations in many cases during last 15 years, see the pioneering papers [3,4], overview [5], introduction to [6] and two more unexpected examples [7,8]. See also recent papers [13–22] and references therein. The solutions obtained there were represented in the form of *Feynman formula*, i.e. as a limit of a multiple integral as the multiplicity goes to infinity. The Schrödinger equation also belongs to the class of evolution equations, and the same way allows to represent the Cauchy problem solution for it in the form of Feynman and quasi-Feynman integral formulas, see [9,13] and references therein.

The specific feature of the research that is now presented is that we use the shift operators (translation operators, in other terminology) instead of integral operators when constructing the Chernoff function $S(t)$. For this reason the solution of (1) is now represented via a new type of formulas that do not include integrals. Our model example, i.e. the Cauchy problem (1), can be modified in different directions. The example was chosen rather simple (free of distinctive or complicated details) intentionally to let the reader focus on the the main contents of the article i.e. on the presented method.

Definition 1. Let $\mathscr{L}(\mathcal{F})$ be the set of all linear bounded operators in a Banach space $\mathcal{F}$. Let the operator $L : \mathcal{F} \supset Dom(L) \to \mathcal{F}$ be linear and closed. The function G is called *Chernoff-tangent* to the operator L iff:

(CT1) G is defined on $[0, +\infty)$, takes values in $\mathscr{L}(\mathcal{F})$, and the function $t \longmapsto G(t)f$ is continuous for each $f \in \mathcal{F}$.

(CT2) $G(0) = I$, i.e. $G(0)f = f$ for each $f \in \mathcal{F}$.

(CT3) There exists such a dense subspace $\mathcal{D} \subset \mathcal{F}$ that for each $f \in \mathcal{D}$ there exists a limit

$$G'(0)f = \lim_{t\to 0} \frac{G(t)f - f}{t}.$$

(CT4) The closure of the operator $(G'(0), \mathcal{D})$ is equal to $(L, Dom(L))$.

Remark 1. In the definition of the Chernoff tangency the family $(G(t))_{t\geq 0}$ usually does not have a semigroup composition property, which in fact is a reason why we often can find a simple formula for $G(t)$. Each C_0-semigroup $(e^{tL})_{t\geq 0}$ is Chernoff-tangent to its generator L, but if L is a differential operator with variable coefficients then usually we do not have a simple formula for e^{tL}. We should not expect to have such a formula because the Cauchy problem for parabolic equation $[u_t'(t) = Lu(t), u(0) = u_0]$ has the solution $u(t) = e^{tL}u_0$, so finding a formula for e^{tL} is equivalent to finding a formula that solves this Cauchy problem for each $u_0 \in \mathcal{F}$, which is usually not an easy task. However, we can obtain approximations to $e^{tL}u_0$ via the following Chernoff theorem, see [1] for details.

Theorem 1 ([2]). *Let $\mathcal{F}$ and $\mathscr{L}(\mathcal{F})$ be as before. Suppose that the operator $L : \mathcal{F} \supset Dom(L) \to \mathcal{F}$ is linear and closed, and function G takes values in $\mathscr{L}(\mathcal{F})$. Suppose that these assumptions are fulfilled:*

(E) There exists a C_0-semigroup $(e^{tL})_{t\geq 0}$ with the generator $(L, Dom(L))$.

(CT) G is Chernoff-tangent to $(L, Dom(L))$.

(N) There exists such a number $\omega \in \mathbb{R}$, that $\|G(t)\| \leq e^{\omega t}$ for all $t \geq 0$.

Then for each $f \in \mathcal{F}$ we have $(G(t/n))^n f \to e^{tL}f$ as $n \to \infty$ with respect to norm in $\mathcal{F}$ uniformly with respect to $t \in [0, t_0]$ for each $t_0 > 0$, i.e.

$$\lim_{n\to\infty} \sup_{t\in[0,t_0]} \left\| e^{tL}f - (G(t/n))^n f \right\| = 0.$$

Definition 2. If (E), (CT) and (N) hold, then we call G *a Chernoff function* for operator L, in other terminology—that the family $(G(t))_{t\geq 0}$ is *Chernoff-equivalent* to the semigroup $(e^{tL})_{t\geq 0}$, see [9,13] for details. We also call $(G(t/n))^n$ *a Chernoff approximation expression* for e^{tL}.

3. Main result

Remark 2. The main result of the paper is formula (4) proven in Theorem 3. It contains $\lim_{n\to\infty}$. After the limit is taken we obtain the exact solution to Cauchy problem (1). For each fixed n the expression under the limit sign is an approximation to the solution. With growth of n such approximations converge to the exact solution uniformly with respect to $x \in \mathbb{R}^1$ and $t \in [0, t_0]$ for each fixed $t_0 > 0$.

Remark 3. Let us denote the set of all (real-valued and defined on the real line) bounded continuous functions as $C_b(\mathbb{R})$, the set of all bounded functions with bounded derivatives of all orders as $C_b^\infty(\mathbb{R})$, and the set of all bounded, uniformly continuous functions as $UC_b(\mathbb{R})$.

Then $C_b^\infty(\mathbb{R}) \subset UC_b(\mathbb{R}) \subset C_b(\mathbb{R})$, and with respect to the uniform (Chebyshev) norm $\|f\| = \sup_{x\in\mathbb{R}} |f(x)|$ the first inclusion is dense, and the last two spaces are Banach spaces.