Applied Mathematics and Computation 325 (2018) 1–14

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A compact fourth-order in space energy-preserving method for Riesz space-fractional nonlinear wave equations

J.E. Macías-Díaz[a,*], A.S. Hendy[b,c], R.H. De Staelen[d]

[a] Departamento de Matemáticas y Física, Universidad Autónoma de Aguascalientes, Avenida Universidad 940, Ciudad Universitaria, Aguascalientes 20131, Mexico
[b] Department of Computational Mathematics and Computer Science, Institute of Natural sciences and Mathematics, Ural Federal University, ul. Mira. 19, Yekaterinburg 620002 Russia
[c] Department of Mathematics, Faculty of Science, Benha University, Benha 13511, Egypt
[d] Department of Mathematical Analysis, Research group of Numerical Analysis and Mathematical Modelling (NaM^2), Ghent University, Ghent 9000, Belgium

ARTICLE INFO

MSC:
65Mxx
65Qxx

Keywords:
Conservative fractional wave equation
Riesz space-fractional equations
Energy-preserving method
Fractional centered differences
High-order approximation
Stability and convergence analyses

ABSTRACT

In this work, we investigate numerically a nonlinear hyperbolic partial differential equation with space fractional derivatives of the Riesz type. The model under consideration generalizes various nonlinear wave equations, including the sine-Gordon and the nonlinear Klein–Gordon models. The system considered in this work is conservative when homogeneous Dirichlet boundary conditions are imposed. Motivated by this fact, we propose a finite-difference method based on fractional centered differences that is capable of preserving the discrete energy of the system. The method under consideration is a nonlinear implicit scheme which has various numerical properties. Among the most interesting numerical features, we show that the methodology is consistent of second order in time and fourth order in space. Moreover, we show that the technique is stable and convergent. Some numerical simulations show that the method is capable of preserving the energy of the discrete system. This characteristic of the technique is in obvious agreement with the properties of its continuous counterpart.

1. Introduction

The design of energy-preserving finite-difference schemes for nonlinear partial differential equations has been an important topic of research since the early studies by L. Vázquez and coworkers in the 1970s. Many nonlinear partial differential equations are known to posses energy functionals that are preserved under suitable boundary conditions, including models like the Schrödinger, the sine-Gordon and the nonlinear Klein–Gordon equations from relativistic quantum mechanics, just to mention some wave equations of physical relevance. Motivated by this fact, several groups of researchers have developed reliable numerical techniques to approximate the solutions of these and other nonlinear conservative systems as well as the constant energy functionals associated to them. The most notable contributions to the state of the art were the energy-preserving finite-difference methodologies proposed for the Schrödinger [1], the sine-Gordon [2,3] and the nonlinear Klein–Gordon regimes [4]. In fact, those works (among other important papers of those decades) still continue to be

* Corresponding author.
E-mail addresses: jemacias@correo.uaa.mx (J.E. Macías-Díaz), ahmed.hendy@fsc.bu.edu.eg (A.S. Hendy), rob.destaelen@ugent.be (R.H. De Staelen).

https://doi.org/10.1016/j.amc.2017.12.002
0096-3003/

sources of motivation for the numerical investigation of nonlinear wave equations [5]. Later on, these studies were extended to account for more general potential functions. In this way, the investigation of energy- or dissipation-preserving methods was extended to more complicated regimes. At the same time, a solid basis for their design was also formulated by D. Furihata and coworkers in various seminal papers [6,7]. In many senses, these works constitute the formal birth of the discrete variational derivative method, whose use has been widely accepted in the specialized literature [8].

In general, the use of numerical methods that preserve invariants obeys various physical and mathematical reasons, including the need to establish analytically some numerical properties. On physical grounds, it is highly desirable to have at hand reliable numerical techniques that resemble the dynamics of the continuous models of interest. In that sense, the early reports by L. Vázquez and co-authors [4] communicated the physical need to develop methods with both numerical and meaningful physical properties. Those reports have been perhaps some of the first efforts in the investigation of structure-preserving methods for partial differential equations or, as R. E. Mickens calls them [9], dynamically consistent numerical techniques. It is worth mentioning that the development of structure-preserving methods has been a fruitful avenue of research in numerical analysis. For instance, this approach has been followed in the design of general multi-symplectic formulations of partial differential equations where the preservation of both energy and momentum is crucial [10], in the design of energy-conserving finite-difference schemes for the simulation of musical instrument contact dynamics [11], in the approximation of solutions of particular models of mathematical physics such as the "good" Boussinesq equation [12] or the Kolmogorov–Fokker–Planck model [13], in the construction of efficient energy-preserving integrators for oscillatory Hamiltonian systems [14], in the design of multi-consistent algorithms for the multi-species Rosenbluth–Fokker–Planck equation [15], in the proposal of new energy-preserving Birkhoffian multi-symplectic methods for Maxwell's equations with dissipative terms [16], in the development of wavenumber-preserving schemes for solving Maxwell's equations in curvilinear non-staggered grids [17] and in the design of energy/dissipation-preserving methods for partial differential equations using the average vector field method [18,19], just to mention some examples.

It is important to note that the notions of 'structure preservation' or 'dynamic consistency' not only refer to the capability of numerical methods to preserve analogues of physical quantities (like energy, momentum, mass, etc.). In a broader sense, these concepts also refer to the capacity of a computational technique to preserve mathematical features of the relevant solutions of continuous systems that naturally arise from the physical context of the problem. A typical example is the condition of positivity (or non-negativity) of solutions, which is a natural requirement for problems in which the variables of interest are measured in absolute scales [20,21]. Boundedness is another desirable characteristic in physical problems where there are natural limitations of growth, particularly in models that describe the dynamics of populations under limited resources and space [22,23] or transport phenomena in turbulent flows. Another mathematical feature of some solutions is the monotonicity, which is important in the approximation of equations whose solution is a cumulative distribution of probability [24] or some traveling waves [25]. In the present work, however, we will consider a nonlinear conservative wave equation with Riesz space-fractional derivatives for which some positive energy functional is preserved under suitable boundary conditions. Motivated by the early works by L. Vázquez [2] and D. Furihata [6], we will design a structure-preserving method that conserves the energy of the system. More concretely, our approach will be based on the use of fractional centered differences, and we will provide discrete schemes for both the solution of the problem and the total energy of the system. We will show here that the total energy of the discretized system is likewise a positive function of the time. The preserved quantities will be used then to show that the method proposed in this manuscript is not only consistent but also stable and convergent of second order in time and fourth order in space. Some simulations will show the capability of the method to preserve the energy under the analytic conditions derived in this work.

Beforehand, we must clarify that the literature also has reports of methods for partial differential equations with fractional derivatives that do not necessarily preserve the structure of the solutions. Indeed. most of the methods proposed are numerically efficient techniques. For example, some numerical methods have been proposed to solve the Fokker–Planck equation with space-fractional derivatives [26], some computational techniques have been used to approximate the solutions of Riesz fractional advection–dispersion equations [27]. Other numerical methods have been designed for the variable-order fractional advection-diffusion equation with a nonlinear source term [28], to solve the multi-term time-fractional wave-diffusion equation [29], to approximate effectively the time-space fractional Fokker–Planck equation with a nonlinear source term [30] and to solve the space- and time-fractional Bloch–Torrey equation through an implicit, stable and convergent numerical method [31]. As a conclusion, many reports in the literature show that the development of numerical techniques to solve fractional partial differential equations (both parabolic and hyperbolic) has been a fruitful avenue of research, but very few reports have striven to design structure-preserving techniques for those systems. The advantage of the technique that we will propose in this work lies in the fact that the preservation of the energy provides a physical meaningfulness to our results. For instance, some physical applications of our methodology may be proposed to various nonlinear phenomena, included the process of nonlinear supratransmission in fractional systems and chains with long-range interactions [32,33].

This manuscript is structured as follows. The nonlinear conservative wave equation with Riesz space-fractional derivatives that motivates our investigation is presented in Section 2, together with the relevant definitions of the fractional differential operators and an energy functional proposed in the literature [34]. We show therein that the initial-boundary-value problem under investigation is a conservative system under suitable analytic conditions. Section 3 introduces the discrete nomenclature and the method to solve numerically the problem under investigation. The concept of fractional centered differences will be recalled therein and some useful lemmas will be proved in the way. The most important physical properties of the method will be established in that section. Concretely, we will establish the capability of the finite-difference scheme to