Applied Mathematics and Computation 325 (2018) 31–40

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A unified framework for asymptotic and transient behavior of linear stochastic systems

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ARTICLE INFO

Keywords:
Stochastic systems
Mean square $(\gamma\alpha)$-stability
Asymptotic and transient behavior
Matrix inequality

ABSTRACT

This paper is concerned with a unified framework for asymptotic and transient behavior of stochastic systems. In order to explain this problem explicitly, a concept of mean square (γ, α)-stability is first introduced and two stability criteria are derived. By utilizing an auxiliary definition of mean square $(\gamma$, T)-stability, the relations among mean square (γ, α)-stability, mean square $(\gamma$, T)-stability and finite-time stochastic stability are established. Subsequently, two new sufficient conditions for the existence of state and output feedback mean square (γ, α)-stabilization controllers are presented in terms of matrix inequalities. A numerical algorithm is given to obtain the relation between $\gamma_{\min}$ and α. Finally, an example is given to illustrate our results.

1. Introduction

It is well-known that Lyapunov asymptotic stability(LAS) presents steady state behavior of systems and is very useful for practical applications. However, an asymptotic stable system may have large values of the states. Often these large values can destroy the system, which are not acceptable. For example, large transient voltage will destroy power systems [1]. To deal with these unacceptable transient values, a concept of finite-time stability(FTS) was proposed in [2], which mainly concerns the transient behavior of the system over a fixed finite-time interval and has been widely extended to deterministic linear continuous-time systems [3–5], discrete-time systems [6], stochastic systems [7,8], Markov jump systems [9–14] and so on. Nevertheless, up to now, there is no a unified framework to consider both transient performance and steady state performance.

On the other hand, following the development of stochastic differential equation, Itô stochastic systems have received much attention because of their practical applications, such as signal processing [15], gene networks [16], mathematical finance [17], epidemic model [18,19]. Therefore, the related stability and control problems have been extensively studied. For example, [20] investigated mean square stability of linear stochastic systems by spectrum technique. In the sequel, literature [21] investigated input-to-state stability for stochastic nonlinear systems with state-dependent switching. [22] investigated stochastic $\mathcal{H}_2/\mathcal{H}_\infty$ control of mean-field type for continuous-time systems with state-and disturbance-dependent noise. For

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https://doi.org/10.1016/j.amc.2017.12.023

some results of robust control on this kind of systems, we refer to the monograph [23]. However, most of the results were only concerned with either transient performance or steady state performance.

Motivated by aforementioned discussions, the unified stability and stabilization problems for asymptotic and transient behavior of stochastic systems are considered in this paper. By using stochastic analysis technology, the stability criteria and some stabilizing conditions are obtained. The contributions of this paper lie in the following three aspects: (1) A concept of mean square (γ, α)-stability is first introduced to put asymptotic and transient behavior of Itô stochastic systems into a unified framework, and two stability criteria are obtained. (2) By utilizing an auxiliary definition of mean square (γ, T)-stability, the relations among (γ, α)-stability, (γ, T)-stability and FTS are established. (3) On the basis of stability criteria obtained, two kinds of unified stabilization controllers (state and output feedback controllers) are designed to make closed-loop systems have the desired transient and steady performance, and a numerical algorithm is given to obtain the relation between $\gamma_{\min}$ and α.

This paper is organized as follows: Section 2 gives a mean square (γ, α)-stability and its relation to mean square (γ, T)-stability and FTS. Section 3 provides several stability conditions for mean square (γ, α)-stability and mean square (γ, T)-stability and analyzes the relations among these stability conditions. Section 4 is to design state and output feedback mean square (γ, α)-stabilization controllers. A numerical algorithm is given in Section 5. Section 6 employs an example to illustrate the results. Section 7 gives the conclusion.

Notations: X^T stands for transpose of a matrix X. The $X > 0$ means that X is positive-definite. $I_{n\times n}$ stands for $n \times n$ identity matrix. $\lambda_{\max}(X)(\lambda_{\min}(X))$ represents the maximum(minimum) eigenvalue of a matrix X. $\mathbb{E}[\cdot]$ denotes the operator of the mathematical expectation. $tr(X)$ is the trace of a matrix X. The notation $\|x\| := \sqrt{x^T x}$ denotes Euclidian 2-norm of a vector x and $\|x\|_Q$ denotes $\sqrt{x^T Q x}$. The asterisk "*" represents the symmetry term in a matrix and diag$\{\cdots\}$ represents a block-diagonal matrix. The shorthand "wrt" is an abbreviation of "with respect to". $k(X) = \lambda_{\max}(X)/\lambda_{\min}(X)$ stands for condition number of a $X > 0$.

2. Preliminaries and definitions

Consider the following linear Itô stochastic system

$$\begin{cases} dx(t) = Ax(t)dt + A_1 x(t)dw(t), \\ x(0) = x_0 \in \mathcal{R}^n, \end{cases} \tag{1}$$

where $x(t) \in \mathcal{R}^n$ is the system state, A, $A_1 \in \mathcal{R}^{n\times n}$ are constant matrices and x_0 is the initial state. Without loss of generality, we assume $w(t)$ to be one-dimensional standard Wiener process defined on the probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ with $\mathcal{F}_t = \sigma\{w(s) : 0 \le s \le t\}$.

Next, a definition of the unified description for asymptotic behavior and transient behavior of system (1) is given.

Definition 1. Given the constants $\gamma > 1$ and $\alpha > 0$, system (1) is called mean square (γ, α)-stable if for a given $x_0 \in \mathcal{R}^n \backslash \{0\}$,

$$\mathbb{E}[\|x(t)\|^2] < \gamma \mathbb{E}[\|x_0\|^2] e^{-\alpha t}, \ t \ge 0. \tag{2}$$

In Definition 1, γ and α are pre-specified. The α embodies asymptotic behavior of the system (1) and the γ embodies the upper bound of transient behavior of the system (1). Definition 1 contains both asymptotic behavior and transient behavior of the system (1).

Remark 1. Definition 1 is different from exponential mean square stability (EMS-stability) in [23]. The EMS-stability is as follows: a system is called EMS-stable if there exist constants $d_1 > 0$ and $d_2 > 0$ such that $\mathbb{E}[\|x(t)\|^2] \le d_1 \mathbb{E}[\|x_0\|^2] e^{-d_2 t}$, $\forall t \ge 0$, for all $x_0 \in \mathcal{R}^n$. The EMS-stability only requires the *existence* of constants $d_1 > 0$ and $d_2 > 0$, which may result in large transient behavior (if γ is large) and small decay velocity (if α is small) of systems. In other words, the existing d_1 and d_2 may not be what we need in practical applications. In addition, for a given (γ, α), an EMS-stable system may not be mean-square (γ, α)-stable.

Next, a proposition equivalent to Definition 1 is given.

Proposition 1. *For the given constants $\gamma > 1$ and $\alpha > 0$, the stochastic system (1) is mean square (γ, α)-stable if and only if*

$$tr(Z(t)) < \gamma tr(Z(0)) e^{-\alpha t}, \ t \ge 0,$$

where $Z(t) > 0$ is the solution to

$$\begin{cases} \dot{Z}(t) = AZ(t) + Z(t)A^T + A_1 Z(t) A_1^T, \\ Z(0) = x(0)x^T(0). \end{cases} \tag{3}$$

Proof. Let $Z(t) = \mathbb{E}[x(t)x^T(t)]$, we obtain

$$\mathbb{E}[\|x(t)\|^2] = \mathbb{E}[tr(x(t)x^T(t))] = tr(\mathbb{E}[x(t)x^T(t)]) = tr(Z(t)). \tag{4}$$

According to (4), it can be obtained that

$$\gamma \mathbb{E}\|x(0)\|^2 e^{-\alpha t} = \gamma tr(Z(0)) e^{-\alpha t}. \tag{5}$$