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A finite element method for Maxwell polynomial chaos Debye model

Changhui Yao[a], Yuzhen Zhou[a], Shanghui Jia[b,*]

[a] School of Mathematics and Statistics, Zhengzhou University, Zhengzhou City 450001, PR China
[b] School of Statistic and Mathematics, Central University of Finance and Economics, Beijing 100081, PR China

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ABSTRACT

'In this paper, a finite element method is presented to approximate Maxwell–Polynomial Chaos(PC) Debye model in two spatial dimensions. The existence and uniqueness of the weak solutions are presented firstly according with the differential equations by using the Laplace transform. Then the property of energy decay with respect to the time is derived. Next, the lowest Nédélec–Raviart–Thomas element is chosen in spatial discrete scheme and the Crank–Nicolson scheme is employed in time discrete scheme. The stability of full-discrete scheme is explored before an error estimate of accuracy $O(\Delta t^2 + h)$ is proved under the L^2–norm. Numerical experiment is demonstrated for showing the correctness of the results.

1. Introduction

For the time propagation of electromagnetic fields, Maxwell–Debye model contains Maxwells equations and polarization models in simple Debye medium, which describes the material response to the fields. Such derivation tacitly is assumed that one has similar individual (molecular, dipole, etc.) parameters, that is, all dipoles, molecules, electron clouds, etc., have the same relaxation parameters, plasma frequencies, etc. Historically, such these models have often not performed well when trying to compare models with experimental data [1,18]. In order to account for uncertainty in the polarization mechanisms, we allow for a distribution of relaxation parameters, and subsequently refer to these types of materials as polydispersive. Maxwell–Random Debye model [11] can be derived in terms of a distribution-dependent dielectric response function. One needs an efficient method for the computation of the expected value of the random polarization. Therefore, a spectral method in random space can be employed, generally called polynomial chaos [19–21], and the model is written by Maxwell–PC Debye.

In the time domain simulation, the applications of time-domain finite element methods(TDFEM) for dispersive media was initialed in [2]. The discontinuous Galerkin methods(DG) for modeling dispersive media was developed in [3]. An interior penalty discontinuous Galerkin method for the time-dependent Maxwells equations in cold plasma was presented in [4] and the error analysis was carried out. The Crank–Nicolson scheme to approximate the electric field equation by Nédélec edge elements was developed and proved to be optimal convergent in energy norm in [5]. FEMs for the time-harmonic Maxwells equations in non-convex polygon [6] was explored. They also investigated TDFEM for the cole-cole dispersive medium model, which contains Maxwells equations and a fraction time derivative term [7]. The mathematical

* Corresponding author.
E-mail addresses: chyao@lsec.cc.ac.cn (C. Yao), 1182519712@qq.com (Y. Zhou), shjia@lsec.cc.ac.cn (S. Jia).

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model was established which described the electromagnetic interrogation of dielectric materials in [8]. At the same time the well-posedness of the system and the regularity of the solution was addressed.

The finite-difference time-domain methods (FDTD) [9–12] is employed frequently. Yee's scheme was proved firstly in [13]. For dispersive media by FDTD were examined in [14–17]. In order to avoid the difficulty of implementing the Cole–Cole model, a representation of dispersive mechanisms using distributions of parameters within the Debye model was presented in [11,18]. Then modify the standard FDTD scheme to allow for distributions of dielectric parameters [24] in a polarization model, and approximate the solution by using the Polynomial Chaos expansion method. The first polynomial chaos for the Random Debye model was considered in [23]. An inverse problem formulation to determine the distribution of parameters was formulated in [11].

In this paper, we recall the 2D Maxwell–PC Debye model and present the existence and uniqueness of the weak solution according with the differential equations by using the Laplace transform. By define the energy norm, we can derive a property of energy decay with respect to the time. Then, we give the stability of the Crank–Nicolson full-discrete scheme. Furthermore, the error estimate of accuracy $O(\Delta t^2+h)$ is proved under the L^2-norm. Finally, numerical experiment is present for showing the correctness of the results.

The rest of this paper is organized as follows. In Section 2, we firstly recall Maxwell–Random Debye model, which is based on the relaxation time τ modeled as a random variable. And Maxwell–PC Debye model is stated by using numerical approximation of Polynomial Chaos expansion. We also prove the well-posedness of the weak formula with respect to Maxwell–PC Debye model by using the Laplace transform, and the property of energy decay is identified, too, after defining the energy norm. In Section 3, we present the Crank–Nicolson full discrete scheme with Nédélec–Raviart–Thomas element to approach the weak formula of Maxwell–PC Debye model and the property of energy decay in discrete form is proved. In Section 4, we provide the error estimates between the discrete scheme and the weak formula. In Section 5, a numerical example is set up to verify the convergence with a small parameter.

2. Recalling Maxwell–PC Debye model

In this section, we will introduce the source of the Maxwell–PC Debye Model. The existence and uniqueness of the solution and the properties of energy decay are given.

2.1. Maxwell–Random Debye model

Firstly, we define some useful two dimensional function spaces as follow

$$H(curl,\Omega)=\left\{\mathbf{u}=(u_1,u_2)\in(L^2(\Omega))^2; curl\mathbf{u}=\frac{\partial u_2}{\partial x}-\frac{\partial u_1}{\partial y}\in L^2(\Omega)\right\},$$
$$H_0(curl,\Omega)=\{\mathbf{u}\in H(curl,\Omega);\mathbf{n}\times\mathbf{u}=0,\text{ on }\partial\Omega\},$$

where in the 2D, we define $\mathbf{n}\times\mathbf{u}:=n_1u_2-n_2u_1$.

A two dimensional transverse electric Maxwell–Random Debye problem can be described by Gibson [11]: finding $\mathbf{E}\in C(0,T;H_0(curl,\Omega))\bigcap C^1(0,T;L^2(\Omega)^2), H\in C^1(0,T;L^2(\Omega))$, $\mathbf{P}\in C^1(0,T;L^2(\Omega)^2)$, $\mathcal{P}\in C^1(0,T;(L^2(D)\otimes L^2(\Omega)^2)$ such that

$$\frac{\partial H}{\partial t}=-\frac{1}{\mu_0}curl\mathbf{E},\qquad in\quad (0,T]\times\ \Omega, \tag{1}$$

$$\epsilon_0\epsilon_\infty\frac{\partial\mathbf{E}}{\partial t}=rotH-\frac{\partial\mathbf{P}}{\partial t},\qquad in\quad (0,T]\times\ \Omega, \tag{2}$$

$$\tau\frac{\partial\mathcal{P}}{\partial t}+\mathcal{P}=\epsilon_0\epsilon_d\mathbf{E},\quad in\quad (0,T]\times\ (D\otimes\Omega), \tag{3}$$

where ϵ_0, ϵ_∞, μ_0 are the dielectric parameters, the electric permittivity and the magnetic permeability, respectively. ϵ_d is defined by $\epsilon_d=\epsilon_s-\epsilon_\infty$ and ϵ_s is the static permittivity. Ω is a polygonal domain in $\mathbb{R}^2$ with boundary $\Gamma:=\partial\Omega$, and $\mathbf{n}$ is the unit outward normal to $\partial\Omega$. In the Maxwell's equations, $\mathbf{E}$, H are the electric fields and magnetic fields, respectively. We also denote $rotH=(\frac{\partial H}{\partial y},-\frac{\partial H}{\partial x})$. $\mathbf{P}$ is the dielectric polarization, and $\mathcal{P}$ is the random polarization due to the random relaxation parameter $\tau\in D=[\tau_a,\tau_b]\subset(0,\infty)$.

The system (1)–(3) is supplemented with the boundary condition

$$\mathbf{n}\times\mathbf{E}=0\qquad on\ \partial\Omega, \tag{4}$$

and initial conditions

$$\mathbf{E}(\mathbf{x},0)=\mathbf{E}_0(\mathbf{x}),H(\mathbf{x},0)=H_0(\mathbf{x}),\mathcal{P}(\mathbf{x},0)=\mathcal{P}_0(\mathbf{x}), \tag{5}$$

for the given function $\mathbf{E}_0(\mathbf{x}),H_0(\mathbf{x}),\mathcal{P}_0(\mathbf{x})$.