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A class of compact boundary value methods applied to semi-linear reaction–diffusion equations☆

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ABSTRACT

This paper deals with a class of compact boundary value methods (CBVMs) for solving semi-linear reaction–diffusion equations (SLREs). The presented CBVMs are constructed by combining a fourth-order compact difference method (CDM) with the p-order boundary value methods (BVMs), where the former is for the spatial discretization and the latter for temporal discretization. It is proven under some suitable conditions that the CBVMs are locally stable and uniquely solvable and have fourth-order accuracy in space and p-order accuracy in time. The computational effectiveness and accuracy of CBVMs are further testified by applying the methods to the Fisher equation. Besides these research, we also extend the CBVMs to solve the two-component coupled system of SLREs. The numerical experiment shows that the extended CBVMs are effective and can arrive at the high-precision.

1. Introduction

Consider the following initial-boundary value problem of SLREs:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = a\dfrac{\partial^2 u}{\partial x^2}(x,t) + g(u(x,t)), & (x,t) \in [0,l]\times[t_0,T], \\ u(x,t_0) = \varphi(x), & x \in [0,l], \\ u(0,t) = \psi_1(t), \quad u(l,t) = \psi_2(t), & t \in [t_0,T], \end{cases} \tag{1.1}$$

where x, t denote the spatial and temporal variables, respectively, $a > 0$ is the diffusion coefficient, $\varphi : [0,l] \to \mathbb{R}$, $\psi_1 : [t_0,T] \to \mathbb{R}$, $\psi_2 : [t_0,T] \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are some given sufficiently smooth mappings. This type of problem plays an important role in modeling real phenomena arising in physics, chemistry, biology and many other scientific fields (see e.g. [1–4]).

Generally speaking, it is difficult to obtain the exact solution of an initial-boundary value problem of SLREs. Hence ones turn to develop various numerical methods to solve the problem. Up to now, a lot of numerical methods for SLREs have been presented. For example, finite difference methods, implicit-explicit predictor-corrector methods, linearized compact multi-splitting methods, compact finite difference methods, spectral methods and continous/discontinous Galerkin finite element

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methods have been introduced and studied in Refs. [4–12], respectively. More related research also refer to the references therein.

Besides the above full discretization methods, in recent years, many researchers considered the so-called method of lines (MOLs) for various partial differential equations (PDEs), where the spatial and temporal discretization methods were combined using. In particular, BVMs and the block BVMs (BBVMs) (cf. [13–24]) are often used as the discretization approximation in time. With this idea, Sun and Zhang [25] introduced the high-order CBVMs for one-dimensional heat equations, Dehghan and Mohebbi [26] studied the high-order CBVMs for unsteady convection-diffusion equations and Liu et al. [27] combined Galerkin–Chebyshev spectral method with BBVMs to derive a class of high-effective MOLs for two-dimensional semi-linear parabolic equations.

The above research show that BVMs are a kind of good candidates in temporal discretization. However, they were not applied to the problem of SLREs. In view of this, in the present paper, we give an investigation to this topic. The paper is organized as follows. In Section 2, by combining a fourth-order compact difference method with BVMs, a class of CBVMs are derived for SLREs (1.1). In Section 3, the local stability and unique solvability of the induced CBVMs are studied and the corresponding criteria are established. In Section 4, the error analysis is performed for the CBVMs and a convergence result is obtained. In Section 5, the CBVMs are applied to the Fisher equation, whose numerical results show that the CBVMs are effective and can arrive at high-precision. In Section 6, the CBVMs are extended successfully to solve the two-component coupled system of SLREs. Finally, in Section 7, a concluding remark is presented to summarize the whole paper and propose some related open problems for the future research.

2. A class of compact boundary value methods

For solving the initial-boundary value problems (1.1), in this section, we will present a class of numerical methods by combining CDMs with BVMs, where the former is for the discretization of spatial variable and the latter for the discretization of temporal variable.

Let $0 = x_0 < x_1 < \cdots < x_m = l$ be a uniform mesh with $x_i = x_0 + ih,\ i = 0, 1, \dots, m,\ h = l/m$, and $\mathcal{W} = \{\omega_i |\ 0 \le i \le m\}$ be the grid function space defined on $\Omega_m = \{x_i |\ 0 \le i \le m\}$. Write

$$\delta_x^2\omega_i = \frac{1}{h^2}(\omega_{i+1} - 2\omega_i + \omega_{i-1}), \quad \mathcal{K}\omega_i = \frac{1}{12}(\omega_{i+1} + 10\omega_i + \omega_{i-1}), \quad i = 1, \dots, m-1. \tag{2.1}$$

On the plane $x = x_i$, the Eqs. (1.1) become

$$\frac{\partial u}{\partial t}(x_i, t) = a\frac{\partial^2 u}{\partial x^2}(x_i, t) + g(u(x_i, t)), \quad i = 1, \dots, m-1.$$

Acting the operator $\mathcal{K}$ on both sides of the above equations yields

$$\mathcal{K}\frac{\partial u}{\partial t}(x_i, t) = a\mathcal{K}\frac{\partial^2 u}{\partial x^2}(x_i, t) + \mathcal{K}g(u(x_i, t)), \quad i = 1, \dots, m-1. \tag{2.2}$$

For giving a spatial discretization scheme, we introduce a result from Numerov [28] (see also Sun [29]).

Lemma 2.1. *(cf. [28, 29]) Suppose $v(x) \in C^6([x_{i-1}, x_{i+1}])$. Then*

$$\frac{1}{12}[v''(x_{i+1}) + 10v''(x_i) + v''(x_{i-1})] - \frac{1}{h^2}[v(x_{i+1}) - 2v(x_i) + v(x_{i-1})] = \frac{h^4}{240}v^{(6)}(\zeta_i), \quad \zeta_i \in (x_{i-1}, x_{i+1}).$$

It follows from (2.1), (2.2) and Lemma 2.1 that there exist a series of bounded functions $r_i(t)$ $(i = 1, \dots, m-1)$ on the interval $[t_0, T]$ such that

$$\mathcal{K}\frac{\partial u}{\partial t}(x_i, t) = a\delta_x^2 u(x_i, t) + \mathcal{K}g(u(x_i, t)) + h^4 r_i(t), \quad i = 1, \dots, m-1. \tag{2.3}$$

A combination of boundary conditions in (1.1) and (2.3) generates that

$$Ky'(t) = aLy(t) + Kf(y(t)) + q(t) + h^4 r(t), \quad t \in [t_0, T], \tag{2.4}$$

where

$$y(t) = (u(x_1, t), u(x_2, t), \dots, u(x_{m-1}, t))^T, \quad f(y(t)) = (g(u(x_1, t)), g(u(x_2, t)), \dots, g(u(x_{m-1}, t)))^T,$$

$$K = \frac{1}{12}(10I_{m-1} + S), \quad L = \frac{1}{h^2}(-2I_{m-1} + S), \quad r(t) = (r_1(t), r_2(t), \dots, r_{m-1}(t))^T,$$