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The Riemann problem for the shallow water equations with horizontal temperature gradients

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ABSTRACT

We consider the Riemann problem for the system of shallow water equations with horizontal temperature gradients (the Ripa system). The model under investigation has the form of a nonconservative system, and it is hyperbolic, but is not strictly hyperbolic. We construct all solutions of the Riemann problem. It turns out that there may be up to three distinct solutions. A resonant phenomenon which causes the colliding shock waves is observed, where multiple waves associated with different characteristic fields propagate with the same shock speed.

1. Introduction

In this paper we consider the Riemann problem for the Ripa system, which was introduced in [27,28] to model ocean currents. The Ripa model was derived from the Saint–Venant system of shallow water equations, in which the horizontal water temperature fluctuations are taken into account. The governing equations of the Ripa model are given by

$$\begin{aligned} &\partial_t h + \partial_x (hu) = 0,\\ &\partial_t (hu) + \partial_x \left(hu^2 + \frac{g}{2} h^2 \theta \right) = -gh\theta \partial_x a,\\ &\partial_t (h\theta) + \partial_x (hu\theta) = 0, \quad x \in I\!R,\ t > 0. \end{aligned} \tag{1.1}$$

Here, h, u, and θ denote the water depth, the depth-averaged horizontal velocity, and the potential temperature field, respectively; g is the gravitational constant, and a is the bottom topography.

Since the system may suffer shock waves, the right-hand side of (1.1) has the form of a *nonconservative* source term. Motivated by earlier works [21,26,34], we supplement the system (1.1) with the trivial equation

$$\partial_t a = 0, \quad x \in I\!R,\ t > 0. \tag{1.2}$$

The system (1.1) and (1.2) can be written as a system of balance laws in nonconservative form, whose weak solutions can be understood in the sense of nonconservative product, see [13,22].

Recall that the Riemann problem for (1.1) and (1.2) is the Cauchy problem with the initial data of the form

$$U(x,0) = \begin{cases} U_L, & \text{for} \quad x < 0,\\ U_R, & \text{for} \quad x > 0, \end{cases} \tag{1.3}$$

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where U_L and U_R are given constant states. The Riemann problem for the usual shallow water equations with discontinuous topography was investigated in [24], where there are two unknowns h and u, and the wave curves are defined in the (h, u)-plane and the Riemann problem can be solved relying on the intersection of these curves. The larger system (1.1) has three unknowns h, u and θ, and wave curves are usually defined in a three-dimensional space. This makes it hard to determine the intersection of wave curves to solve the Riemann problem.

In addition, the Rieman problem for the model of a fluid flow in a nozzle with discontinuous cross-section was considered in [34]. Although the model in [34] also has three unknowns, it has a very nice property as the one in the usual gas dynamics equations: across the contact discontinuities $\lambda = u$, the two quantities u and p remain constant. This allows us to project wave curves into the (p, u)-plane to determine the intersection of appropriate wave curves, which leads to solutions of the Riemann problem. However, the model (1.1) possesses a different property: across the contact discontinuities $\lambda = u$, only u remains constant. This raises a very hard problem for determining the intersection of wave curves to solve the Riemann problem. To deal with this very challenging problem, we introduce in the present work a new kind of composite wave curves: the curves of composite waves between waves associated with a nonlinear characteristic field and the linearly degenerate field $\lambda = u$. These composite wave curves together with the curves of composite waves between the other nonlinear field and the linearly degenerate field $\lambda = 0$ of the system (1.1) and (1.2), and the curves of single waves will play the key role in solving the Riemann problem for the system.

There have been many works in the literature for the study on nonconservative systems of balance laws. Early works can be found in [16,17,21,26]. The Riemann problem for nonconservative models was considered in [3,14,23,24,34]. Nonconservative systems and two-phase flow models were studied in [5,7,18,22]. The Riemann problem for two-phase flow models was studied in [4,33,35]. Numerical schemes for the Ripa system were constructed in [8,15,31,36]. Godunov-type and van Leer-type schemes were presented in [1,9–12,25,30,32]. Other numerical schemes for nonconservative models were considered in [2,6,19,20,29]. See also the references therein.

The organization of this paper is as follows. Section 2 is devoted to basic properties of the model, where the hyperbolicity, nonstrict hyperbolicity, genuine nonlinearity and linear degeneracy of the characteristic fields, shock waves, rarefaction waves and contact discontinuities are studied. In Section 3 we will study to select admissible stationary contact waves. The Riemann problem for (1.1)-(1.2) is considered in Section 4 for both cases of a flat and of a non-flat bottom.

2. Basic properties and elementary waves

2.1. Hyperbolicity and genuine nonlinearity

We first derive the system (1.1) and (1.2) in terms of the primitive variables $U = (h, u, \theta, a)$. To simplify the notations, in what follows we use

$$(.)_t = \partial_t(.), \quad (.)_x = \partial_x(.).$$

For smooth solutions, the first equation of (1.1) can be written as

$$h_t + uh_x + hu_x = 0.$$

The second equation of (1.1) can be expanded by

$$uh_t + hu_t + huu_x + u(hu)_x + \frac{gh^2}{2}\theta_x + gh\theta h_x + gh\theta a_x = 0.$$

The first and the fourth terms of the last equation cancel each other, due to the first equation of (1.1). Then, divide both side of the last equation by $h > 0$ to get

$$u_t + g\theta h_x + uu_x + \frac{gh}{2}\theta_x + g\theta a_x = 0.$$

The third equation of (1.1) can be expressed as

$$h\theta_t + \theta h_t + hu\theta_x + \theta(hu)_x = 0,$$

which sees the second and the fourth terms liquid each other, by the first equation of (1.1). Then, we divide both sides of the last equation by $h > 0$ to get

$$\theta_t + u\theta_x = 0.$$

Thus, we can see that the system (1.1) and (1.2) can be written as a system of balance laws in nonconservative form

$$U_t + A(U)U_x = 0,$$

where $U = (h, u, \theta, a)^T$, and